Foundations of Neuroimaging
Martin Sereno
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Vector Add, Multiply
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Gradient Echo
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Fourier Transform
Slice Selection
Frequency Encoding
Phase Encoding
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Hardware

**MAGNET HARDWARE**

1. $B_0$ field from superconducting magnet
2. Gradient coils
3. Body RF transmit/receive
4. RF receive-only
5. Shim coils (in gradient)

$B_0 \to z$ (longitudinal)

$B_1 \to x, y$ (transverse)

(1) $B_0$ field

(2) Body gradient coils

(3) RF transmit body coil

(4) RF receive-only head coils

max gradient:

$80 \text{ mT/m}$

$200 \text{ T/m/sec}$

\( I_T = 10,000 \text{ Gauss} \)

Earth: 0.25-0.65 G

25 - 65 mT

\( \frac{Y}{2\pi} = 42 \text{ MHz/T} \)

RF transmitter (30 kW)

RF receiver

Circularly polarized $B_1$ field

Rotating $B_1$ to $B_0$ at Larmor freq.

$B_1$ is several orders of magnitude smaller than $B_0$

Three 1.5 million watt amplifiers to add ramps to $B_0$ field

Shim coils also embedded in here (not shown)

Z

X

Y

\( \sim \) non-superconducting water-cooled, external shield

Supercconducting coils in liquid helium

(No power required after current injected to bring up field using induction)
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei with odd atomic weight or odd proton numbers).
- Moving charge creates a magnetic field.

![Classical Picture](image)

- Current loop from spinning charge (right-hand rule).
- N.B.: Classically this would cause EM radiation, spindown.

**Stern-Gerlach Experiment**

Pass silver atoms through a strong magnetic field → split into just 2 beams.

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**Microscopic Picture**

- If no strong magnetic field, \( B_\phi = 0 \)
- Strong magnetic field, \( B_\phi \) (up/down)
- Strong \( B_\phi \) plus oscillating \( B_1 \)

**Precession**

- Distinguish precession (slow) from spin (fast).
- Treat classically, like spinning top.

\[
2\pi I_0 \frac{B_\phi}{L} = \frac{\omega_0}{\gamma} = \hbar \frac{B_0}{\gamma}
\]

- Larmor frequency (eg. 63 MHz)
- Gyro-magnetic ratio (eg. 1.57)

- Precessing vectors are "bunched" at any one moment around circle.

**Macroscopic Picture**

- Bulk magnetization:
  
  \[
  M = \frac{1}{4} K T S
  \]
  
  Where \( I = \pm \frac{1}{2} \), \( \frac{I(I+1)}{2} \), and \( \frac{1}{4} K T S \)

- Two non-constants:
  
  \[
  B_0 \rightarrow \text{i.e., } M_z^0 \text{ proportional to } B_\phi \text{ strength}
  \]
  
  \[
  N_0 \rightarrow \text{i.e., } M_z^0 \text{ proportional to number spins}
  \]

- \( K \) = Boltzmann const.

- \( T_0 \) = Abs. temperature sample.

- Precession like tops: precession faster w/ more gravity.

N.B.: Compared to tops, gravity & relaxation:

- Frictionless spin, doesn’t slow.
- Signed gravity.
- Can change precession dir.
- Can stick under floor.
- Neighbor bumping causes decay (1/2).
**Bloch Equation**

- Time-dependent behavior of \( \vec{M} \) in the presence of an applied magnetic field (excitation \& relaxation).

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B} - \frac{\vec{M} \times \vec{B}}{T_2} - \frac{M_x \hat{x} + M_y \hat{j}}{T_1} - \frac{(M_z - M_z^0) \hat{k}}{T_1}
\]

In the Larmor-rotating coordinate system, a tilt \& a phase shift in a standard \( B_1 \) excitation is rotation around \( x \)-axis.

- Longitudinal and transverse relaxations

\[
\frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_z^0}{T_1}
\]

\[
\frac{dM_x' y'(t)}{dt} = -\frac{M_x' y'(t)}{T_2}
\]

- Solution to equations above: time-dependent free precession eq's.

Given initial \( M_x', M_y' \):

- Rotating frame:
  - \( M_z'(t) = M_z^0 \left( 1 - e^{-t/T_1} \right) + M_z'(0) e^{-t/T_1} \)
  - \( M_x'(t) = M_x'(0) e^{-t/T_2} \)

Lab frame: same!

- Re-growth from 0 after pulse-decaying:
  - \( M_z'(1) = 63\% M_z^0 \)
  - \( M_x'(1) = 37\% M_x'(0) \)

Lab frame: times \( e^{-\omega t} \)
Vector Add, Multiply

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers
  - Generalizes to any \( D \)

\[ \|\vec{c}\| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

**Dot Product**

(= inner product)

(= "scaled projection onto")

\[ c = \vec{a} \cdot \vec{b} = \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z \]

- Scalar

\[ p = \|\vec{a}\| \cos \theta \]

\[ c = p \|\vec{b}\| \]

- Generalizes to any \( D \)

**Cross Product**

(= outer product)

(= can be generalized: see "geometric algebra")

\[ \vec{c} = \vec{a} \times \vec{b} = \begin{bmatrix} 0 & -b_x & b_y \\ b_x & 0 & -b_z \\ -b_y & b_z & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \]

- Vector

Geometric algebra: bivector plane area

- Right-hand rule: curl fingers from \( \vec{a} \) to \( \vec{b} \); thumb is \( \vec{c} \)

- Unique orthogonal specific to 3D

- \( \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta \)

- Max if orthogonal

**Complex Multiply**

(see also quaternions, geometric algebra generalization)

\[ \vec{c} = \vec{a} \cdot \vec{b} = \begin{bmatrix} b_x & -b_y \\ b_y & b_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} a_x b_x - a_y b_y \\ a_x b_y + a_y b_x \end{bmatrix} \]

- Vector

\[ \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \]

- Angles add

- Magnitudes multiply

- Specific to 2D

\[ \text{Sum of angles: } \theta_1 + \theta_2 \]

\[ \text{Like real nums} \]
EFFECTS OF $\hat{M}$, $\vec{B}$, and $\Theta$ ON PRECESSION FREQ.

Bloch 1st term: $\frac{d\hat{M}}{dt} = \hat{M} \times \vec{YB}$

cross prod. properties review:

$\left\| \frac{d\hat{M}}{dt} \right\| = \left\| \hat{M} \right\| \left\| \vec{YB} \right\| \cdot \sin \Theta$

Starting condition:

$\Rightarrow$ now see effects of changing $\hat{M}$, $\vec{B}$, $\Theta$

Change $\hat{M}$ length:

$\Rightarrow \frac{d\hat{M}}{dt}$ proportionally larger, so canals

$\Rightarrow$ same precession freq. as starting cond.

Change $\Theta$ between $\hat{M}$ and $\vec{B}$:

$\Rightarrow \frac{d\hat{M}}{dt}$ goes up (then down) as $\sin \Theta$

but circumference also goes up as $\sin \Theta$,

$\Rightarrow$ same precession freq.

Change $\vec{B}$ length:

$\Rightarrow \frac{d\hat{M}}{dt}$ goes up, proportional to $\vec{B}$

but circumference is same at starting cond.

$\Rightarrow$ increased precession freq. ($\omega = \gamma \vec{B}$)
**Simple Matrix Operations**

**Basic Idea**
- A matrix \( \begin{bmatrix} \text{rotates/scales} \end{bmatrix} \) a vector
  \[ \vec{b} = M \vec{a} \]

**3D Example**
- \( \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \)

**Add Translate (after rotate/scale)**
- Commonly used "hack" for aligning rows
  - A 4D matrix \( \begin{bmatrix} \text{rotates/scales} \\ \text{then} \end{bmatrix} \) a 3D vector
    \[ \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]
  - N.B.: Have to keep track of order!!
    - Rotate/scale then trans ≠ trans, then rot/scale
    - Change rot component: untranslate, rot, retranslate

**3 Special Cases (3D):** Rotate around each major axis without changing length
- Rotate around \( x \)-axis:
  \[ R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \]
  - E.g., 90° flip

- Rotate around \( y \)-axis:
  \[ R_y(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \]
  - E.g., 180° flip to avoid add 180° phase after 90° flip on \( x' \)

- Rotate around \( z \)-axis:
  \[ R_z(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
  - E.g., precession with B\( \phi \) along \( z' \)

**General Case**
- Rotate around general \( z' \)-axis:
  \[ R_{\phi}(\alpha) = R_z(-\Theta)R_y(-\Phi)R_z(\alpha)R_y(\Phi)R_z(\Theta) \rightarrow \text{(quaternions are more efficient)} \]
SOLUTIONS TO SIMPLE DIFFERENTIAL EQ.

diff. eq.: \[ \frac{dM_{xy}(t)}{dt} = -\frac{M_{xy}(t)}{T_2} \]

Solution: \[ M_{xy}(t) = \frac{M_{xy}(0)}{T_2} \cdot e^{-t/T_2} \]

Goal: 1) find \( M_x \) whose derivative satisfies diff. eq.
2) also find \( M_x \) (one of many) that passes thru init condition

\( \Rightarrow \) try exponential, since derivative \( (e^x) = e^x \)

<table>
<thead>
<tr>
<th>diff. eq.</th>
<th>soln.</th>
<th>deriv. ( \frac{dM}{dt} )</th>
<th>check</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M(t) = \frac{-1}{T_2} \cdot M(t) )</td>
<td>( e^{-t/T_2} )</td>
<td>( -\frac{1}{T_2} \cdot \frac{M(t)}{T_2} )</td>
<td>OK - we have recovered our diff eq.</td>
</tr>
<tr>
<td>( M(t) = \frac{0.5}{T_2} \cdot e^{-t/T_2} )</td>
<td>( -\frac{1}{T_2} \cdot \frac{M(t)}{T_2} )</td>
<td>( \frac{1}{T_2} \cdot \frac{M(t)}{T_2} )</td>
<td>N.B.: this function is the &quot;unknown&quot; like the ( x ) in ( x + 1 = 3 )</td>
</tr>
</tbody>
</table>

Another soln: \( M(t) = \text{const} \cdot e^{-t/T_2} \)

<table>
<thead>
<tr>
<th>deriv. ( \frac{dM}{dt} )</th>
<th>check</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\frac{1}{T_2} \cdot \text{const} \cdot e^{-t/T_2} )</td>
<td>( -\frac{1}{T_2} \cdot \frac{M(t)}{T_2} )</td>
</tr>
</tbody>
</table>

\( \Rightarrow \) any const \( \text{OK!} \)

Initial condition:
const = \( M_{xy}(0) \)

\( M'(t) = \frac{M_{xy}(0)}{T_2} \cdot e^{-t/T_2} \)

Magnetization immediately after RF (Bl) ends.
VERIFY SOLUTION TO T1 REGROWTH

- Slightly more complex T1 soln compared to T2 soln

**T2 soln verify (from prev)**

**T1 solution verify**

\[
\frac{dM}{dt} = \frac{M_{xy}}{T_2}
\]

**Original diff eq.**

\[
M'(t) = \frac{-1}{T_2} \cdot M(t)
\]

**Make unknown funct M(t) more visible**

\[
M(t) = \begin{cases} M_{xy}(0_t) e^{-t/T_2} & \text{Proposed solution} \\ \end{cases}
\]

\[
M'(t) = \begin{cases} \frac{-1}{T_2} M_{xy}(0_t) e^{-t/T_2} & \text{Test by take deriv.} \\ \end{cases}
\]

\[
\frac{dM}{dt} = \frac{-(M_z - M_z^0)}{T_1}
\]

**Init cond.**

\[
M(t) = \begin{cases} M_z^0 \left(1 - e^{-t/T_1}\right) + M_z(0_t) e^{-t/T_1} & \text{chain rule as before} \\ \end{cases}
\]

\[
M'(t) = \begin{cases} M_z^0 \left(-e^{-t/T_1} + M_z(0_t) e^{-t/T_1}\right) & \text{chain rule} \\ \end{cases}
\]

- Derivative in original T1 eq. says \(M(t)\) minus \(M_z^0\)

\[
M'(t) = \frac{-1}{T_1} \left( M(t) - M_z^0 \right)
\]

Solution \(\left[M_z^0 - M_z^0 e^{-t/T_1} + M_z(0_t) e^{-t/T_1}\right]\)

- Which equals our re-calculated derivative:

\[
M'(t) = \frac{-1}{T_1} \left( -M_z^0 e^{-t/T_1} + M_z(0_t) e^{-t/T_1} \right)
\]
**Bloch Eq. - Matrix Version**

\[
d\vec{M}/dt = \vec{M} \times \gamma \vec{B}_\phi
\]

Solution:
\[
\vec{M}(t) = \left[ \begin{array}{c} M_x(t) \\ M_y(t) \\ M_z(t) \end{array} \right] = \left[ \begin{array}{ccc} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} M_x(0_t) \\ M_y(0_t) \\ M_z(0_t) \end{array} \right] = R_z(\omega t)M(0_t)
\]

Include Relaxation:
\[
d\vec{M}/dt = \vec{M} \times \gamma \vec{B}_\phi - \frac{M_x I + M_y J}{T_2} - \frac{(M_z - M_z^0)K}{T_2}
\]

Solution:
\[
\vec{M}(t) = \left[ \begin{array}{ccc} e^{\gamma t/2} & 0 & 0 \\ 0 & e^{\gamma t/2} & 0 \\ 0 & 0 & e^{\gamma t/2} \end{array} \right] \left[ \begin{array}{ccc} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} M_x(0_t) \\ M_y(0_t) \\ M_z(0_t) \end{array} \right] + \left[ \begin{array}{c} 0 \\ 0 \\ M_z(0_t)(-e^{\gamma t/2}) \end{array} \right]
\]
**Excitation in the Rotating Frame**

- **Original Bloch e.q. in laboratory frame**
  \[
  \frac{\text{d} \mathbf{M}}{\text{d}t} = \mathbf{M} \times \mathbf{B} + \mathbf{B}_\text{f}
  \]

  - Add on-resonance \( B_1 \) to \( \mathbf{M} \)
  \[
  \mathbf{B} = B_1(t) \left( \cos \omega_0 t \mathbf{\hat{z}} - \sin \omega_0 t \mathbf{\hat{j}} \right) + B_\text{f} \mathbf{\hat{k}}
  \]

  - Matrix version
  \[
  \frac{\text{d} \mathbf{M}}{\text{d}t} = \begin{bmatrix}
  \frac{\text{d} M_x}{\text{d}t} \\
  \frac{\text{d} M_y}{\text{d}t} \\
  \frac{\text{d} M_z}{\text{d}t}
  \end{bmatrix} = \begin{bmatrix}
  0 & \omega_0 & \omega_1(t) \\
  -\omega_0 & 0 & \omega_1(t) \cos \omega_0 t \\
  \omega_1(t) \sin \omega_0 t & -\omega_1(t) \cos \omega_0 t & 0
  \end{bmatrix}
  \begin{bmatrix}
  M_x' \\
  M_y' \\
  M_z'
  \end{bmatrix}
  \]

  - Substitution to convert to the rotating frame
  \[
  \mathbf{M}' = \mathbf{R}_x(\omega_0 t) \cdot \mathbf{M}_{\text{rot}}
  \quad \mathbf{B}' = \mathbf{R}_x(\omega_0 t) \cdot \mathbf{B}_{\text{rot}}
  \]

  - After substitution any off-resonance appears as residual \( B_\text{f} \)
    \( (B_2) \)
    (see off-res notes page)

  - Rotating frame < on-resonance
    * basic excite, \( B_1 \)-only, no gradient
    \( \rightarrow \) removes \( \omega_0, \cos/\sin \)

  - Rotating frame < off-resonance
    * general, \( B_1 \)-only incl gradients
    \[
    \frac{\text{d} \mathbf{M}_{\text{rot}}}{\text{d}t} = \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & \omega_1(t) \\
    0 & -\omega_1(t) & 0
    \end{bmatrix}
    \begin{bmatrix}
    M_x' \\
    M_y' \\
    M_z'
    \end{bmatrix}
    \]

  - Gradient: \( \omega(x) = \gamma G_z x \)
    off-res: appears as residual \( B_\text{f} \), lifting \( B_1 \) vect.
    out of \( x-y \) plane

    - This means \( \mathbf{M} \) vect. update will contain
      component that rotates \( \mathbf{M} \) around
      \( z \)-axis (in rotating coords \( z \)-axis)

    - Small tip: \( \frac{\text{d} \mathbf{M}_{\text{rot}}}{\text{d}t} \approx
      \begin{bmatrix}
      0 & 0 & 0 \\
      0 & \omega_1(t) & 0 \\
      0 & 0 & \omega_1(t)
      \end{bmatrix}
      \begin{bmatrix}
      M_x' \\
      M_y' \\
      M_z'
      \end{bmatrix}
      \]

      - Zeros this line in matrix
      \( \rightarrow \) small tip \( \Rightarrow \) easier to solve!
Bloch Eq. Summary

\[
\frac{d\hat{M}}{dt} = \hat{M} \times \hat{\mathbf{B}} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_z^0) \hat{k}}{T_1}
\]

(Lab-frame)

(Vector lengths not to scale!)

- Full lab-frame picture is complex:
  - 3 component of \( \frac{d\hat{M}}{dt} \) update vector
  - Larmor freq. component 7-9 orders magnitude larger than \( T_2 \), \( T_1 \) decay
  - \( \hat{B}_1 \) is also rapidly wiggling

- Conceptual simplification in 4 stages:

1) **Lab frame**
   - Just precession

2) **Rotating frame**
   - \( \hat{M} \) stopped
   - That is, \( \hat{B}_0 = 0 \)

3) **Add \( \hat{B}_1 \)**
   - \( \hat{B}_1 \) also stopped!
   - But \( \hat{M} \times \hat{B} \) still works!
   - "Precess" around \( \hat{B}_1 \) axis

4) **Off-resonance**
   - Slow precess, now around tilted \( \hat{B}_{eff} \)

Signal eq.
RF coil only picks up \( M_x \), \( M_y \).
RF FIELD POLARIZATION

- Polarization (change of direction)
- Linearly polarized field
  \[ \vec{B}_1(t) = B_1 \cdot \cos \omega t \hat{x} \]
  Magn. strength: \{1, \frac{1}{2\pi}, \frac{1}{\sqrt{2}}\}

- N.B.: \( \vec{B}_1 \) adds to much larger \( \vec{B}_0 \)

- Circularly polarized field (quadrature)
  \[ \vec{B}_{1c}(t) = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \]
  \[ = B_1 \cdot e^{-i\omega t} \hat{z} \]
  Rotates

- Flat quadrature coil

- In the rotating coordinate system, flipping around x-axis vs. y-axis
  is just difference in phase of RF

- Typical 90° flip (around x-axis)
- Typical 180° flip (around opposite y-axis)

180° flip

90° power at 90°
**SIGNAL EQUATION**

\[ \Phi(t) = \int_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- Magnetic flux through coil (integral of magnetic field perpendicular to area)

- Local magnetization by coil geometry of object (time-dependent)

- Time deriv. inside: use Bloch process, see deriv. cos \( \Rightarrow \) sin deriv. sin \( \Rightarrow \) cos

- Evaluate using free precession eg. (solution to Bloch) ignoring relaxation

- Ignore change in z-comp. \( \mathbf{M} \) because slow \( \Rightarrow \) i.e., we only see \( M_{xy} \), not \( M_z \)

- Substitute \( \mathbf{M}(t) \) with lab frame \( M_{xy}(t) = M_{xy}(0) e^{-\gamma w t} \)

- Simplify:
  1. Ignore decay (assume this \( t=0 \))
  2. Assume phase-sensitive detection

\[ \mathbf{S}(t) = \int_{\text{obj}} M_{xy}(\mathbf{r}, 0) e^{-i \mathbf{S}_w(\mathbf{r}) t} \, d\mathbf{r} \]

- Spatially-dependent resonant freq in rotating frame \( \Rightarrow \) i.e. after subtraction \( \mathbf{S}_w \)

- Standard signal expression

- Phase angle in rotating frame

- At a single time point, RF signal is vector sum across object of local transverse magnetization vectors
**PHASE-SENSITIVE DETECTION**

- **V(t) → multiply → Low-Pass Filter → S(t)**
  - $V(t) \approx 123$ MHz
  - $S(t) \approx 50$ kHz
  - method for moving very high frequency Larmor oscillations down to tractable frequency range

Demodulated signal $\propto$ RF coil signal $\cdot$ reference (transmitter)

$\propto \sin[(\omega_0 + \delta\omega)t] \cdot \sin[\omega_0 t]$

- $\sin a \sin b = \frac{1}{2} [\cos(a-b) - \cos(a+b)]$
- $\sin a \cos b = \frac{1}{2} [\sin(a-b) + \sin(a+b)]$

$\frac{1}{2} [\cos \delta\omega t - \cos (2\omega_0 + \delta\omega)t]$

- This signal is digitized
- Filter this one out with low pass filter

**One freq → freq domain**

- Signal
- Reference
- Demodulated
- After filter (rotating frame)

**Chirp → time domain**

- Chirp
- Center
- Demodulated
- No freq signal
- Filter
- $t \rightarrow \Rightarrow$ phase!

- Two signals are made from a single receiving RF coil
- A quadrature coil can be treated the same way (OK to combine after adding $\frac{\pi}{2}$ phase, then PSD)
- Quadrature coil has better S/N since noise in each part is uncorrelated ($\frac{1}{\sqrt{2}}$ better)
FID - FREE INDUCTION DECAY, $T_2^*$

- Signal (FID) resulting from RF pulse w/angle $\alpha$

$$S(t) = \sin \alpha \sqrt{\rho(w) \cdot e^{-t/T_2(w)} \cdot e^{-i\omega t}} \, dw$$

 recorded complex signal

- An example spectral density ("Lorentzian inhomogeneity")

$$\rho(w) = \frac{M_0^2}{(Y AB\phi)^2 + (w - w_0)^2}$$

- Complex signal

$$\tilde{S}(t) = \frac{\pi \cdot M_0^2 \cdot Y AB\phi \cdot \sin \alpha \cdot e^{-t/2} \cdot e^{-T_2/2} \cdot e^{-i\omega t}}{\text{from integral of } \frac{\Delta w}{\Delta w + w_2}}$$

- N.B. center freq., not original integration variable

$$\frac{1}{T_2} = \frac{1}{T_2^*} + \frac{1}{T_2^\prime}$$

- $T_2^*$ overall decay rate including inhomogeneous $\phi$
Spin Echo, Stimulated Echo

$90^\circ - \tau - 180^\circ, \frac{T_2}{T_2^*} \neq \text{echo}$

**Rotating Coords**

- Just after $90^\circ x'$ pulse $f_{x'} + f_{y'}$ have same phase
- Relaxation + phase dispersion of $f_{x'} + f_{y'}$
  (both from $B > B_0$)
- Just after $180^\circ y'$ pulse
  ($y'$ pulse like $x'$ pulse but RF has $+90^\circ$ phase)
  echo caused by re-phasing of $f_{x'} + f_{y'}$
  (w/ decay due to $T_2$)
- Remember brief RF just tips vectors while retaining length
  relaxation includes tips and shrinks ($M_T$) and grows ($M_z$ echo)
- $180^\circ x'$ pulse works, too, but echo will have $+\pi$ phase (left side in figs above)
- Echo generated even if second pulse not $180^\circ$ (see next)

**FID decay (and echo growth/decay)**
  described by $T_2^*$; from inhomogeneity

- Reduction in height of echo compared to initial described by $T_2$
  echo fixes "star"
**ECHOES — Spin echo**

\[ \alpha_1 - T - \alpha_2 - T \]  
(both pulses along \( y' \) for simplicity)

**Effect of \( \alpha_y \) pulse**

\[
\begin{align*}
M_x' &= M_x \cos \alpha - M_z \sin \alpha \\
M_y' &= M_y \\
M_z' &= M_x' \sin \alpha + M_z' \cos \alpha
\end{align*}
\]

\( \text{etc. for } \alpha_x, \alpha_2 \)

**Effect of \( T \) delay**

\[
\begin{align*}
M_x' &= (M_x' \cos \omega T + M_y' \sin \omega T) e^{-\gamma T/2} \\
M_y' &= (-M_x' \sin \omega T + M_y' \cos \omega T) e^{-\gamma T/2} \\
M_z' &= M_z^0 (1 - e^{-\gamma T}) + M_z' e^{-\gamma T/2}
\end{align*}
\]

**Immediately after \( \alpha_1 \) pulse**

\[
\begin{align*}
M_x'(w, 0_+^i) &= -M_z^0 (w) \sin \alpha \, , \\
M_y'(w, 0_+^i) &= 0 \\
M_z'(w, 0_+^i) &= M_z^0 (w) \cos \alpha
\end{align*}
\]

**For one isochromat of freq. \( w \)**

**After \( T \) delay**

\[
\begin{align*}
M_x'(w, T) &= -M_z^0 (w) \sin \alpha, \cos \omega T e^{-\gamma T/2} \\
M_y'(w, T) &= M_z^0 (w) \sin \alpha, \sin \omega T e^{-\gamma T/2} \\
M_z'(w, T) &= M_z^0 (w) [1 - (1 - \cos \alpha_1) e^{-\gamma T/2}]
\end{align*}
\]

**Immediately after \( \alpha_2 \) pulse (no effect on \( M_y' \); rewrite \( y_+^i \); combine \( x \) and \( y \) etc.)**

\[
\begin{align*}
M_x'(w, T) &= M_z^0 (w) \sin \alpha, \left( \sin^2 \frac{\alpha_2}{2} e^{i \omega T} - \cos \frac{\alpha_2}{2} e^{i \omega T} \right) e^{-\gamma T/2} \\
&\quad - M_z^0 (w) \sin \alpha, \sin \omega T e^{-\gamma T/2}
\end{align*}
\]

**Time dependent free precession around \( z' \)**

(rewrite \( M_y'(w, T) \))

\[
M_x'(w, t) = M_x'(w, T) e^{-(t-T)/T} e^{-i \omega(t-T)}
\]

\[
= M_z^0 (w) \sin \alpha, \sin \frac{\alpha_2}{2} e^{-t/T} e^{-i \omega(t-2T)} \\
- M_z^0 (w) \sin \alpha, \cos \frac{\alpha_2}{2} e^{-t/T} e^{-i \omega T}
\]

\[
- M_z^0 (w) [1 - (1 - \cos \alpha_1) e^{-\gamma T/2}] \sin \alpha \sin \frac{\alpha_2}{2} e^{i \omega(t-2T)} - i \omega T
\]

**For a large num of freq's:**

[terms 2, 3 are dephasing \( \rightarrow \) FID of echo]  
[term 1 is nphasing \( \rightarrow \) nphase at \( t = 2T \)]

\[
\begin{align*}
S_1(t) &= \sin \alpha_1, \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/T} e^{-i \omega(t-T)} d\omega \\
A_E &= \sin \alpha_1, \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-TE/T} d\omega
\end{align*}
\]

\[\rho(w) = \frac{\sin \alpha, \sin^2 \frac{\alpha_2}{2}}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/T} e^{-i \omega(t-T)} d\omega\]

\[\rho(w) = \frac{\sin \alpha, \sin^2 \frac{\alpha_2}{2}}{2} \int_{-\infty}^{\infty} \rho(w) e^{-TE/T} d\omega\]

\[\rho(w) = \frac{\sin \alpha, \sin^2 \frac{\alpha_2}{2}}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T} d\omega\]

\[\text{echo amplitude, ignoring} \]

\[\text{freq. dependence of } T^2\]

\[\text{etc. for } A_E \ldots \text{ like } \]

**Peak and echo signal from 1**

\[S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/T} e^{-i \omega(t-T)} d\omega\]

\[S_2(t) = \text{no } \frac{1}{2} \text{ factor}\]

\[S_2(t) = \text{multiply by } i \rightarrow \text{add } \frac{1}{2} \text{ phase}\]
**Echo TRAINS** - spin-echo trains

- It's (too) easy to make echoes...

\[ E_n = \frac{3^{n-1} - 1}{2} \]

Echos after end of nth pulse
3 RFs \( \Rightarrow \) 4 echoes (here)
6 RFs \( \Rightarrow \) 121 echoes (!)

Secondary echo: \( SE_{i,2} \) acts like RF pulse
\( \alpha_3 \) makes an echo from it

Two more conventional two pulse spin echoes

- A useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2\( \tau \) spacing

Stimulated echo: combined effect of 3

\[ \alpha_1: M_L \rightarrow M_T \]
\[ \alpha_2: \text{leftover } M_T \text{ flipped to } M_L \text{ (saved)} \]
\[ \alpha_3: \text{flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays (after being held in limbo between 180°, } FID_2 \text{ and } FID_3); \text{ acts like 2-pulse echo} \]

- Typically, 90° and 180° applied in different axes (\( x', y', z' \)) which reduces phase errors due to imperfect 180° pulses (since slightly-off rotation around \( y' \) affects phase less)
- Using full Bloch eq. solutions is tedious 😞
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize $90^\circ, 180^\circ$)
- Problem #1: $\alpha$ pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into $M_L$
- Problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

$\Rightarrow$ Rule for effect of $\alpha$
RF pulse on transverse mag

$\Rightarrow$ Rule for effect of $\alpha$
RF pulse on longitudinal mag

$\Rightarrow$ Echo when phase path crosses zero
3 - Pulse Echo Amplitudes

- Assume $M_z^0 = 1$

RF transmit

RF receive

<table>
<thead>
<tr>
<th>Echo</th>
<th>Time</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SE_{1,2}$</td>
<td>$t = 2T_1$</td>
<td>$\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2\gamma_1 T_2}$</td>
</tr>
<tr>
<td>$SE_{1,3}$</td>
<td>$t = 2(T_1 + T_2)$</td>
<td>$\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2\gamma_1 T_2}$</td>
</tr>
<tr>
<td>$SE_{2,3}$</td>
<td>$t = T_1 + 2T_2$</td>
<td>$\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-\gamma_1 T_2} e^{-2\gamma_1 T_2}$</td>
</tr>
<tr>
<td>STE ($&quot;$stimulated$&quot;$)</td>
<td>$t = 2T_1$</td>
<td>$1 - (1 - \cos \alpha_1) e^{-\gamma_1 T_1} \sin \alpha_2 \sin^2 \frac{\alpha_3}{2} e^{-(T_1 + 2T_2)/T_2}$</td>
</tr>
<tr>
<td>$SE_{1,3}$</td>
<td>$t = 2(T_1 + T_2)$</td>
<td>$\sin \alpha_1 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2(T_1 + T_2)/T_2}$</td>
</tr>
</tbody>
</table>

- $T_1$-dependence in STE (but also $SE_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse
**Gradient Echoes** - T₂*, GE chains

- Initial negative gradient dephases spins.
- After t = T₀ of positive gradient, spins rephase.
- Does not correct for T₂* inhomogeneities.
  - So echo amplitude is
    \[ A_E = e^{-t/T_2*} \]
  - The initial "FID" is not "free" since it is being actively de-phased by gradient, so FIDdecay.

- Key difference between spin-echo (SE) and gradient echo (GE) is that B₀ inhomogeneities not canceled.
  - Hence, echoes are T₂*-weighted, not T₂-weighted → more susceptible to inhomogeneities.

- Echo trains possible w/ gradient echo (CPMG-like)
  - The faster the gradients are switched, the more echoes you get.
  - EPI hardware → 64 echoes.
**IMAGE CONTRAST**

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

![RF pulses diagram]

- Simple saturation/recovery w/ no echo
- Initial conditions:
  \[ M_z \text{ before first pulse} = M_z^0 \]
  \[ M_z = 0 \text{ immediately after first pulse (i.e., 90° pulse) \[ M_z \text{ before second pulse} \]}

- From Bloch eq, \( M_z \) just before second pulse:
  \[ M_z^{(n)}(O_-) = M_z^0 \left(1 - e^{-TR/T1} \right) + M_z^{(n)}(O_+) e^{-TR/T1} \]

- Given:
  1. 90° pulse
  2. no \( M_{xy} \) left

  → Pure tip: \( M_{xy} = M_z \)

- Tip existing magnetization:
  \[ M_z^{(n)}(O_-) = M_{xy}(O_+) = M_z^0 \left(1 - e^{-TR/T1} \right) \]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[ I(r) = C \rho(r) \left(1 - e^{-TR/T1(r)} \right) \]

\( I(r) \) is the reconstructed spectral density, \( \rho(r) \) is the density of the underlying tissue, and \( T1(r) \) is the relaxation time.
**IMAGE CONTRAST**

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
- because they are brighter than all the rest
- because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur (e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state for typical 1-2 sec TR images reached after n/8 images
**IMAGE CONTRAST**

IR (still just saturation-recovery — no echo)

- Inversion recovery w/ no echo

**RF**

- 180° pulse reverses longitudinal magnetization
  \[ M_x'(0) = -M_x^0 \]

- Recovery to end of first TI from long. part of Bloch eq.
  \[ M_x'(0) = M_x^0 \left( 1 - 2e^{-\frac{TI}{T_1}} \right) \rightarrow \text{flipped into transverse by second pulse (180°)} \]

- Longitudinal then regrows from zero
  \[ M_x'(0) = M_x^0 \left( 1 - e^{-\frac{(TR-TI)}{T_1}} \right) \]

- After second 180°, just change sign again
  \[ M_x'(0) = -M_x^0 \left( 1 - e^{-\frac{(TR-TI)}{T_1}} \right) \]

- Apply relaxation eq. again
  \[ M_x'(0) = M_x^0 \left( 1 - e^{-\frac{TI}{T_1}} \right) - M_x^0 \left( 1 - e^{-\frac{(TR-TI)}{T_1}} \right) e^{-\frac{TI}{T_2}} \]

\[ M_x'(0) = M_x^0 \left( 1 - 2e^{-\frac{T_1}{T_1}} + e^{-\frac{TR}{T_1}} \right) \]

\[ \rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
27

**IMAGE CONTRAST**

- Steady state mag (2nd TR) just before 90°
  \[ M_z^0 (2) = M_z^0 (1 - 2e^{-TR/TE/2}/T1 + e^{-TR/T2}) \]

- The echo signal \( M_z^0 \) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation.

\[
A_E = M_z^0 (1 - 2e^{-TR/TE/2}/T1 + e^{-TR/T2}) e^{-TE/T2}
\]

- If we assume TE much less than TR, then we can simplify:

\[
A_E = M_z^0 (1 - e^{-TR/T1}) e^{-TE/T2}
\]

- Similar equation for SE-IR

\[
A_E = M_z^0 (1 - 2e^{-TI/T1} + e^{-TR/T2}) e^{-TE/T2}
\]

- N.B. centering G_x also causes gradient echo.
**IMAGE CONTRAST**

GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch eq., again
  - assume $M_1^{(n)}(O_-) = 0$ → transverse dephased before next pulse
  - $M_2^{(n)}(O_-) = M_2^0 (1 - e^{-TR/T1}) + M_2^{(n-1)}(O_+) e^{-TR/T1}$ → long TR or spoiler

- Assume we have a small tip angle:
  - $M_2 \cos \alpha \Rightarrow M_2^{(n)}(O_+) = M_2^{(n)}(O_-) \cos \alpha$
  - $M_2^{(n)}(O_-) = M_2^0 (1 - e^{-TR/T1}) + M_2^{(n-1)}(O_-) \sin \alpha e^{-TR/T1}$ → steady state

- Assume we are in dynamic equilibrium:
  - $M_2^{(n)}(O_-) = M_2^{(n-1)}(O_-) = M_2^{ss}(O_-)$

prepulse

steady state

longitudinal

post-pulse

transverse magnetization

gradient echo amplitude

$A_E = \frac{M_2^0 (1 - e^{-TR/T1})}{1 - \cos \alpha e^{-TR/T1}} \sin \alpha e^{-TE/T2*}$

$T1$ contrast mainly depends on flip angle, not TR → $\cos \theta = 1$ → eliminates $T1$ weight since denominator = numerator
- Saturate, wait for contrast, invert, wait for contrast, FLASH (cont'd out)

A) \( M_z' \) (just after 90°) = 0 (perfect 90°)

B) \( M_z' \) (after TD) = \( M_z^0 \left(1 - e^{-TD/T2}\right) \) (Bloch term #1)

C) \( M_z' \) (just after invert) = \( \cos \phi M_z^0 \left(1 - e^{-TD/T2}\right) \)

D) \( M_z' \) (after TI) = \( M_z^0 \left(1 - e^{-TI/T2}\right) + \left[\cos \phi M_z^0 \left(1 - e^{-TD/T2}\right)\right] e^{-TI/T2} \)

\[ = M_z^0 \left[1 - \left[1 - \cos \phi \left(1 - e^{-TD/T2}\right)\right] e^{-TI/T2}\right] \]

Special case TI = TD: \( = M_z^0 \left[1 - e^{-TI/T2}\right]^2 \)

\( \Rightarrow \) using hard 180° inversion with small hard alpha B1 inhomogeneities (Thomas et al, '05)

- after the first pulse:

E) \( M_z' \) (just after pulse) = \( M_z^0 \left[1 - \left[1 - \cos \phi \left(1 - e^{-TD/T2}\right)\right] e^{-TI/T2}\right] \sin \alpha \)
MAGNETIZATION TRANSFER CONTRAST

- Protons in macromolecules & bound to membranes have a wide range of resonant freqs ("bound") → T2 = 1 msec
  - i.e., signal that is visible with usual TE

- Free protons in blood, CSF, water have a narrow range of resonant freqs ("free") → T2 = 50 msec

- Mag transfer pulse sequence
  1) Off-center freq pulse to hit "bound" (but don't hit water too hard)
  2) Wait for magnetization transfer from saturated longitudinal M_L of "bound" → M_L of "free"
  3) Result of transfer → attenuation

- N.B. This always happens a little (cf. T1-weighted, T2-weighted)
  Something to keep in mind if hard pulse (wide freq.)

- Used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- View w/ MIP: maximum intensity projection along lines
  → view as movie
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: \( \text{SNR} = \frac{S_{\text{avg}}}{\sigma_{\text{avg}} \text{ obj signal}} \)
- Temporal SNR: \( \text{SNR}_{\text{t}} = \frac{S_{\text{t}}}{\sigma_{\text{t}}} \)
- "Contrast" is a difference
- Contrast-to-noise ratio:

\[
\text{CNR}_{\text{AB}} = \frac{S_{\text{A}} - S_{\text{B}}}{\sigma_{\text{avg}}} \quad \text{e.g., WM-GM activated, rest}
\]

\[
A_{\text{E}} = M_{\text{r}}^2 (1 - e^{-TR/T1}) e^{-TE/T2}
\]

**Gradient echo**:

\[
A_{\text{E}} = \frac{M_{\text{r}}^2 (1 - e^{-TR/T1}) \sin \alpha}{1 - \cos \alpha e^{-TR/T1}}
\]

**General rules**: spin-echo, long TR GE
SIGNAL-TO-NOISE S/N

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \frac{\Delta x \Delta y \Delta z}{\sqrt{N_a N_x N_y N_z \Delta t}}
\]

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  \[\text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}\]

- More voxels (with size of voxels, \(\Delta t\) per read step constant), \(N\)th effect on S/N
  \[\text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{\sqrt{128 \times 128}}{\sqrt{64 \times 64}} = 2 \text{ times better S/N}\]

- # acquisitions, \(\sqrt{N}\) better S/N
  \[\text{e.g., } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{\sqrt{2}}{1} = 1.41 \text{ times better S/N}\]

- Larger timestep during readout, \(\sqrt{\Delta t}\) better S/N

\[\Delta t = \frac{1}{\text{BW}_{\text{read}}}, \text{ digitization timestep during echo acquisition}\]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog low-pass filter
- \(\Delta t\) controls \(\text{BW}\) because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \(\Delta t\)
- Must filter out freq's > \(f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
**COMPLEX ALGEBRA**

### Real/Imaginary
- **add**: \( (r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2) \)
- **mult**: \( (r_1, i_1) \times (r_2, i_2) = (r_1r_2 - i_1i_2, r_1i_2 + i_1r_2) \)

### Angle/Phase
- **add**: \( (A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2 , A_1 \sin \phi_1 + A_2 \sin \phi_2) \)
- **multiply** (commutative): \( (A_1, \phi_1) \times (A_2, \phi_2) = (A_1A_2, \phi_1 + \phi_2) \)
- **divide**: \( (A_1, \phi_1) \div (A_2, \phi_2) = (A_1/A_2, \phi_1 - \phi_2) \)

- Complex to real power: \( (A, \phi)^n = (A^n, n\phi) \)

- **\( e^{i\phi} \)**
  - **Expand as series**
  - **Recognize cos, sin series**
  - The real "e" to "purely imaginary" power
  - \( e^{i\phi} = \cos \phi + i \sin \phi \)
  - \( e^{i\phi} = (\cos \phi + i \sin \phi)^n \)
  - \( = \cos n\phi + i \sin n\phi \)

- **Fourier Transform**
  - \( H(f) = \int h(t) e^{-2\pi ift} dt \)
  - \( H(\mathbf{f}) = \int h(t) e^{-2\pi ift} dt \)

- **Convolution**
  - \( f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) dz \)
  - \( \mathbf{f}(x) \ast \mathbf{h}(x) = \int \mathbf{g}(z) \cdot \mathbf{h}(x-z) dz \)

- **Convolution Theorem**
  - \( \mathcal{F}[g(x) \ast h(x)] = \mathcal{G}(k) \ast \mathcal{H}(k) \)

- For arbitrary amplitude, multiply \( A e^{i\phi} \)

- Phase is integral of freq. variable \( \phi = \int \omega dt \)

- **N.B.**: 3rd kind of vector multi. different than dot product and cross product (and G.A. non-commutative pseudoscalar multiply)

- Shorthand for unit vector \( (\hat{e} \phi) \) pointing in the direction of \( \phi \)

- For arbitrary amplitude, multiply \( A e^{i\phi} \)

- Phase is integral of freq. variable \( \phi = \int \omega dt \)

- Chose in phase is freq \( \frac{d\phi}{dt} = \omega \)

- because FT, fast if kernel not small
Fourier transform (1)

\[ H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i \frac{2\pi ft}{t}} dt \]

- How to calculate \( H(f) \) for one \( f \) (\( f = 3 \)):

(Real signal: only need 2 correlations)

F (real frequency domain)\[ \rightarrow \]

\[ \cos (3t) \rightarrow \sin (3t) \]

\[ e^{-i3t} \rightarrow \int \text{complex multiply} \]

\[ \text{integrate/sum these multiplies across all } t \]

Real frequency domain\[ \rightarrow \]

\[ b \rightarrow \]

Imaginary frequency domain

\[ \text{like correlating with } \sin \text{ and } \cos \text{ (at each freq.) so we get phase (at each freq.)} \]
**Fourier transform (1b)**

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} = \cos(-\phi) + i \sin(-\phi) = \cos \phi - i \sin \phi \]

- \(\cos\) is an "even" function, \(\sin\) is an "odd" function

**An orthogonal decomposition**

- think of discretely sampled \(\sin(bx), \cos(bx)\) as vectors
- \(\text{Corr}(\vec{V}_1, \vec{V}_2) \equiv \) projection of \(V_1\) onto \(V_2 \equiv \vec{V}_1 \cdot \vec{V}_2\)

\[
\begin{align*}
\text{Corr}(\cos bx, \sin bx) &= 0 \\
&= \sin &\text{and cos of same frequency are orthogonal} \\
&= \sin 2x &\text{cos 2x}
\end{align*}
\]

\[
\begin{align*}
\text{Corr}(\sin bx, \sin bx) &= 0 \\
&= \text{different integer freqs of sin (or cos) are orthogonal} \\
&= \sin 2x &\text{sin 3x}
\end{align*}
\]

\[
\text{Corr}(\cos bx, \sin bx) = 0 \\
[\text{as above}]
\]

- in the continuous case, orthogonal functions defined as:
\[
\int_{-\infty}^{\infty} f(x) g(x) \, dx = 0
\]
FOURIER TRANSFORM OF AN IMAGE (2)

1. Real image → Imaginary image → Fourier Transform → Real spatial freq.
   Imag. spatial freq.

2. Amplitude image → Phase image → Inverse Fourier Transform → Amplitude spatial freq.
   Phase spatial freq.

3. Complex vectors → Zero vectors → View complex vectors directly.

- 3 equivalent representations of image & spatial freq. space.
FOURIER TRANSFORM OF REAL IMAGE (2)

- what a single k-space point looks like for real image (polar coordinates $A, \phi$ instead of $r, \theta$)

**image space**

- Offset of stripes is k-space phase
- Brightness of stripes proportional to k-space amplitude

**k-space**

- Distance from center is stripe spacing
- Angle of point perpendicular to angle of stripes

**amplitude**

- (Should be all zero not same as "stripe phase" above)

**phase**

- (image recon.)

**inverse Fourier transform**

- Value from 0 to 360°

---

- Cartesian dimension of k-space — x- and y- spatial freq

**N.B.:** each dimension of spatial freq. space (k-space) from correlation with sin & cos — don’t confuse $k_x, k_y$ with sin, cos!

- x-component of spatial freq (hi freq)
- y-component of spatial freq (lo freq)

**N.B.:** increasing one 1D component increases the spatial freq of the 2D wave and rotates it
FOURIER TRANSFORM OF IMAGE (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)

- example: cosinusoid in image space, then shifted in x-dir

REAL IMAGE

\[ I(x, y) = \cos(x) \]

FT OF REAL IMAGE

\[ \text{FT of } I(x, y) \]

\[ k_x = 1, k_y = 0 \]

\[ k_x = -1, k_y = 0 \]

\[ \text{Real component less than above because rot:} \]

\[ \text{Phase now } 45^\circ \text{ at } k_x = 1, k_y = 0 \]

\[ (-45^\circ \text{ at } k_x = -1, k_y = 0) \]

N.B: an example of the "Fourier Shift Theorem" (see below)

45° rot compared to complex above
FOURIER TRANSFORM OF IMAGE (5)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

\[ I(x, y) = 1 + \cos(x) \]

center of k-space:

\[ H(k) = \int_{-\infty}^{\infty} h(x) e^{-2\pi ikx} \, dx \]

avg image brightness \[ \approx I(\text{real}) \]

FT OF REAL IMAGE

[positive center k-space]

FT

[the center of k-space is zero w/pure sin or cos image b/c avg brightness = 0]

FT \^{-1}

COMPLEX IMAGE

\[ I(x, y) = \cos(x) - i \sin(x) \]

\[ = e^{-ix} \]

FT, FT \^{-1}

[missing spike results in single spike correlating with \cos and \sin]

N.B.: this k-space is non-Hermitian

k-space will only have Hermitian symmetry if image is real:

Hermitian symm. when complex conjugate (complex num w/ sign flipped in image part) is equal to function w/ arg arg:

1D: \[ H(k) = H^*(k) \]

2D: \[ H(-k_x, k_y) = H^*(k_x, k_y) \]

[complex]

FT OF COMPLEX IMAGE

spike only on one side of k-space

N.B.: this is like what an artifact "spike" does tho it would have round, phase

complex [N.B. this is also exactly what a gradient does to image space!]
FOURIER TRANSFORM OF IMAGE (6)

- (cont.) x- and y-spatial freqs.
- special case: real image from sum of reals

REAL IMAGE

I(x,y) = \cos(x) + \cos(y)

FT OF REAL IMAGE

N.B. adds but doesn't rotate stripes

I(x,y) = \cos(x+y)

Remember, single k-space point transforms to complex img.
but if Hermitian symmetry, imaginary components cancel

Since all we want in image space reconstruction is real component, can just add real components of complex vectors at each image space point for every complex image corresponding to each k-space point

N.B: the k-space phase will affect offset of real-valued image space cosinusoid

Therefore for real-valued image, we can visualize inverse FT as real-valued sum of offset real-valued cosinusoids

N.B. cannot do this with MRI k-space data since phase errors (incl. multiple wraps) mess up real component — must use amplitude img
- Gradient coils for x, y, z generate approximately a linear gradient in the strength of the z-component of the magnetic field $B_z$.

- For example, the x gradient coil induces a ramp in the z-component of the magnetic field when moving in the x-direction:

$$B_{G,z} = G_x x$$

*Since a pure linear gradient of $B_{G,z}$ in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction ($B_{G,x}$ and $B_{G,y}$).*

- The other magnetic field components are usually ignored because they are so small relative to $B_0$, since $B_{G,z}$ is added to $B_0$, and since $B_0$ is much stronger than $B_{G,z}$, $B_{G,x}$, and $B_{G,y}$.

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms $B_{G,x}$ and $B_{G,y}$ are known; can be included in the reconstruction process:

$$\Delta \phi G_x(x) \approx \frac{-x^2 G_x^2 t}{2B_0}$$
SLICE SELECTION (G_z)

- slice select gradient on during RF stim

\[ B_z \]

\[ f = \frac{x}{\pi} (B_0 + B_{G_z}) \]

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

\[ \text{in practice, Gaussian pulse envelope good too} \]

- this excites protons in a narrow slab

\[ t \rightarrow \text{Fourier Transform} \rightarrow 6 \]

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing z-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: \( 90^\circ \rightarrow 52\% \)

- in practice: adjust to max, use crusher to kill spurious transverse on \( 180^\circ \)
PULSES FOR SLICE SELECTION

Fourier approach details

- Fourier transform approach to slice-selective pulse (linear approx., even though tipping is non-linear)

\[ \mathbf{B}_1(t) \propto \int_{f=0}^{f=\infty} p(f) \cdot e^{-i 2\pi f t} \, df \]

i.e., time-dependent complex (quadrature) pulse waveform in Fourier transform of frequency spectrum of RF pulse

\[ \mathbf{B}_1(t) = A \cdot f_w \cdot \text{sinc}(\pi f_w t) \cdot e^{-i 2\pi f_c t} \]

- Amplitude controlling flip angle
- Frequency determined by freq. width, \( f_w \) (N.B. wider \( f_w \) is narrower sinc)
- Modulation oscillating at center freq., \( f_c \)

\( f_w \)

\( f_c \)

\( \text{sinc} \)

Larmor oscillating

\[ \text{sinc} \text{ envelope width inversely proportional to } f_w \]

Fourier Transform Pairs, Rules

- Multiplication in one domain equals convolution in other:

\[ F \left[ g(t) \cdot h(t) \right] = G(f) \ast H(f) \]

\( \ast \) means do \( \ast \) \n
- Convolution with delta function impulse moves other function to impulse center
SLICE SELECT RF PULSES

Interleaved Acquisition \[\rightarrow\] better S/N b/c imperfect slice profile

Common RF pulses

non-selective pulse ("hard" pulse)

standard slice select sinc

Gaussian

\[\rightarrow\] pulses need to be "apodized" (have "foot" removed)
\[\rightarrow\] multiply by function so begin/end of pulse is differentiable

Fat Saturation

- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
- need high quality (narrow-freq) pulse to avoid saturate water!

HOWTO

1) fat sat pulse
2) wait T2 so fat signal decays but no T1 regrowth of fat
3) RF stim for water "protons" of interest"

Adding Another Gradient Tilts Slice

- with 3 gradients on, can excite arbitrary angle plane
- translate plane by changing either gradient amplitude or RF freq band: \[\Gamma_{B2}\]
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) chemical shift change freq. → gradient changes freq.</td>
<td></td>
</tr>
<tr>
<td>2) stimulate w/ broadband RF → same</td>
<td></td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs → same</td>
<td></td>
</tr>
<tr>
<td>4) FT of FID to get spectrum of Δf offsets → FT of FID to get Δx offsets</td>
<td></td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading.
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID.

- the "k-space" perspective is a "Copernican turn".
  - idea is that data is not a set of samples of a time-domain signal generated by multiple chemical-shift like frequencies.
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations (which are analogous to multiple time points).
  - i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>Signal</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>spectroscopy</td>
<td>samples of oscillations in time-domain → FT → frequency-domain spectrum of shifts</td>
</tr>
<tr>
<td>MRI</td>
<td>samples of spatial freq. in freq. domain → FT → spatial object (like a time-domain signal)</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because FT ≈ FT⁻¹ (except sign change).
FREQUENCY ENCODING (1)

- Frequency encode gradient \( G_x \) causes precession rates to vary linearly in \( x \)-direction.

\[ \text{precession} \uparrow \quad B_z \uparrow \quad \text{in} \quad x \text{-direction} \]

\[ \Rightarrow \text{correct} \quad \text{(remember that strength of} \ G_x \text{causes variation of slope of} \ B_z \text{in} \ x \text{-direction)} \]

- Different frequency signals are mixed together and recorded as a 1-D signal over time.

\[ \Rightarrow \text{correct}, \text{ but remember, we are recording summed local magnetization vectors after de-modulation} \]

- A Fourier transform, which can convert back and forth between \( x \)-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal.

\[ \Rightarrow \text{correct} \]

- Spatial frequencies get confused/conflicted with precession frequencies.

\[ \Rightarrow \text{wrong} !! \]

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position.

\[ \Rightarrow \text{conceptually wrong} !! \]

\[ \Rightarrow \text{Ft actually converts spatial frequencies to spatial position} \]

\[ \Rightarrow \text{the spatial frequency increases for each time point in the readout} \]

\[ \Rightarrow \text{the precession freq ramp is constant each timestep} \]

N.B.: Gradient freq ramp does not need to be on during recording!!
FREQUENCY ENCODING (2) connect intuition - why phase critical

- "Frequency" - encode gradient ($G_x$) turned on during during echo causes precession rates to immediately vary with x-position

  $G_x \rightarrow t \rightarrow \uparrow B_z$ in x-direction

  $G_x$ levels (= slope)

  actually

- at beginning of gradient on, the phase of signal coming from each x-position is the same summed phase angle is what we measure

- early after gradient on, phase advances (because of faster precession frequency) arise with greatest phase advance at largest x-position

- later during gradient on, phase advances cause multiple wraparounds of phase angle across space

- in practice, the lowest spatial frequency (= 0) occurs in the middle of the gradient on time because the phase is "wound" negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs

  0 is spatial frequency

  $G_x \rightarrow t \rightarrow \uparrow B_z, x$

  0 = max negative

  $B_z = 0$

  $B_z = max positive$

  individual RF data samples (after demodulation)
**FREQUENCY ENCODING (3)** why each datapoint is 1 spatial freq

Standard Fourier transform:  \[ H(\beta) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i2\pi\beta t} dt \] (Temporal freq ↔ time)

*"k" is often used instead of "f" for the frequency variable.*

Imaging equation:  \[ S(\beta) = \int_{-\infty}^{\infty} I(x) \cdot e^{-i2\pi\beta x} dx \] (Spatial freq ↔ space)

- **Signal strength at one x-position** (brightness of image point)
- **Spin density** (spectral density)
- Oscillations come from readout phase wrapping, where \( f \) is single spatial freq (e.g., 5) and \( x \) goes across object.

To make image, do inverse Fourier transform of recorded signal \( S(\beta) \)

- Get single readout point by summing signal across x-position (RF coil records sum)
- Even though variable is \( \beta \), it represents one time point during readout
- Sum across \( x \) of object
- This is done by RF coil recording sum
- Oscillations come from readout phase wrapping, where \( f \) is single spatial freq (e.g., 5) and \( x \) goes across object.

F = G x t, that is, spatial freq depend on amount of time gradient was on (this \( f \) increases with time!)

Don't confuse with instantaneous changed precession freqs which are constant across entire readout time (for each \( x \) position)
**ALTERNATE DERIVATION** (incl. effects of $G_x$) SIGNAL EQ

- oscillators at $w = k_x$ at each position (just $x$ for now)

$$S(t) = m(x) e^{-i \phi(x)} dx$$

- by definition, freq, $w$ is rate of change of phase, $\phi$

$$\frac{d \phi(x,t)}{dt} = w(x,t) = k_x$$

and integrating

$$\phi(x,t) = \int_0^t w(x,t) dt = \int_0^t B(x,t) dt$$

- assuming phase initially 0, $B$ affected by gradients

$$B(x,t) = B_0 + G_x(t) \cdot x$$

so

$$\phi(x,t) = \chi \int_0^t B_0 dt + \left[ \chi \int_0^t G_x(t) dt \right] x$$

$$= w_0 t + 2\pi k_x(t) x$$

$k$ is time integral of gradient waveform

- demodulation removes the $B_0$-caused carrier frequency $e^{-i w_0 t}$ from the first equation

$$S(t) = \int_x m(x) e^{-i 2\pi k_x x} dx$$

amplitude of each oscillator

gradient-controlled phase
PHASE-ENCODE GRADIENT $G_y$

- Turn on gradient after excitation but before readout

- Different levels of $G_y$

- Higher levels of $G_y$ (slope of $B_z$ in y-direction!)
  $\rightarrow$ Higher spatial freq. (more phase wraps) in y-direction

- Phase wraps persist after phase-encode gradient off

- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase

2D Imaging Equation

$$S(k_x, k_y) = \iint \left[ I(x, y) \cdot e^{-i2\pi(k_x x + k_y y)} \right] \, dx \, dy$$

- Signal recorded at single time point (one readout point)

- Complex signal from phase-sensitive detection

- Sum across x-y plane done by RF coil

- Scalar (what we try to reconstruct)

- Phase angle (of scalar magnetization in rotating frame), set by gradients

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia" — they stay wherever the gradients last left them
3-D IMAGING

- Use \( z \)-gradient for 2nd phase-encoding instead of slice selection
- Excitation of whole slab (slice-select is whole brain)
- Simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [3PGR])

\[
S(k_x, k_y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{I}(x, y, z) e^{-i2\pi (k_x x + k_y y + k_z z)} \, dx \, dy \, dz
\]

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

> phase stripes created throughout volume vs. slice:

N.B, this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

- Since the phase-encode gradient and the freq-encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

N.B.: Stripes have sharp edges from phase wrap (not sinusoid since \(\phi\) from 2-comp quadrature!)

- Successive readout steps:

- 3D phase encode w/ \(G_y\) and \(G_z\) starts rotated in y-z plane.

- Large phase encode \(G_y\) and \(G_z\) spins at a point might be 2 cyles ahead while after y-gradient spins at same point 8 cyles ahead, but counting wraps in y-direction still only 2 ahead.
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point

\[ k = \nabla \int G(t) \, dt \]

spatial freq. gradient recorded at t = record time

simple form of integral w/ boxcar gradient

\[ k = G t \]

(k is area under curve)

all of the following gradients end up at the same point in k-space:

**Frequency-encode**

**FID**

**Frequency-encode gradient echo**

**Frequency-encode spin-echo (plus gradient echo!!)**

**Phase-encode then frequency encode gradient echo**

---

**RF** 90°

\[ G_x \]

samples

\[ k_y \]

\[ k_x \]

---

**RF** 90°

\[ G_x \]

\[ K_y \]

\[ k_x \]

\[ \text{neg} \, G_x \]

---

**RF** 90° 180° TE

\[ G_x \]

\[ K_y \]

\[ k_x \]

\[ \text{pos} \, G_x \]

\[ \text{RF} 180° \]

N.B. 180° waves to conjugate point

---

**RF** 90°

\[ G_y \]

\[ G_x + G_y \]

---

\[ k_y \]

\[ k_x \]
\[ S(k_x, k_y) = \iint \frac{I(x, y) e^{-i 2\pi (k_x x + k_y y)}}{2\pi k_x k_y} dk_x dk_y \]

\[ I(x, y) = \iint S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} dk_x dk_y \]

- In practice, finite number of samples, \( N \) and \( M \), are collected. 
- \( k_x \) and \( k_y \) directions of \( k \)-space (integral \( \Rightarrow \) discrete sum)

\[ I(x, y) = \sum_{m=-M/2}^{M/2-1} \left[ \sum_{n=-N/2}^{N/2-1} S(n, m) e^{i \frac{2\pi n \Delta k_x}{\Delta k_x}} e^{i \frac{2\pi m \Delta k_y}{\Delta k_y}} \right] e^{i \frac{2\pi m \Delta k_y}{\Delta k_y}} \]
**SAMPLING**

- Must consider effects of sampling limited points in \( k \)-space
  - limited in range of frequencies sampled \((k_{\min} \rightarrow k_{\max})\)
  - limited in rate of sampling \((\Delta k)\)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

**Spatial Frequency**

- Correct reconstruction
  - plus replicas

- As above with blurring, ringing

- Aliasing occurs in spatial domain
  - replicas overlap, causing wraparound

- Finite frequency range
  - too-wide spacing of samples

- Infinite frequency range
  - infinitely fine sampling

- Insufficient samples in time domain

**space**
UNDER/OVER SAMPLE

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]
\[ \delta_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

Fov (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is FOV divided by K-space sample count

3 more examples (not included: less samples to same spatial freq [bottom last page])

Basic Image

Same num samp. to 2x spatial freq. (i.e., gradients stronger or time ON longer)

2x num. samples to same spatial freq. (i.e., gradients weaker or time ON shorter)

2x number samples to 2x spatial freq. (i.e., gradients stronger or time ON longer)

N=10
\( k_x = 5 \)
\( \Delta k_x = 1 \)
\( \text{FOV} = 1 \)
\( \delta_x = 0.1 \)

N=10
\( k_x = 10 \)
\( \Delta k_x = 2 \)
\( \text{FOV} = 2 \)
\( \delta_x = 0.05 \)

N=20
\( k_x = 5 \)
\( \Delta k_x = 0.5 \)
\( \text{FOV} = 2 \)
\( \delta_x = 0.05 \)

N=25
\( k_x = 10 \)
\( \Delta k_x = 1 \)
\( \text{FOV} = 1 \)
\( \delta_x = 0.05 \)

- Basic image
- Square pix
- X-pix half width
- Replicas intrude
- Scanner makes square image "wrap" occurs
- Square pix
- Twice x-pix count so FOV = 2x
- This is "phase oversamp"
- Scanner crops to square replicas move out
- X-pix half width
- Twice x-pix count
- Same FOV
- This is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. Image/brain space
2. Sampled data spatial frequency

- Convolve
- Multiply

Useful FTs

- Rect: $\text{Rect} \left( \frac{x}{w} \right) \xrightarrow{\mathcal{F}} W \cdot \text{sinc}(\pi W k)$
- Gaussian (special case): $e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2}$
- Gaussian (adj width): $e^{-ax^2} \xrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} e^{-\frac{\pi k^2}{a}}$
- Comb: $\sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta k}) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p \Delta k)$

Limit approach to Fourier transform of comb

$\Delta k = \frac{1}{\text{FOV}}$

$\text{FOV} = 1/\Delta k$
**Point-Spread Function**

\[ \hat{I}(x) = \Delta k \sum_{n} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- set true image to \( S \)-function, then measured signal is:
  \[ S(n \Delta k) = 1 \]

- substitute into recon to get PSF:
  \[ h(x) = \Delta k \sum_{n} e^{i 2\pi n \Delta k x} \]

- simplify
  \[ h(x) = \Delta k \frac{\sin \left( \pi N \Delta k x \right)}{\sin \left( \pi \Delta k x \right)} \Rightarrow \text{periodic} \]

- that is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \( k \)-space is an image sinc

---

**Image**

how PSF modifies ideal (infinite \( k \) image)

\[ \text{convolve} \]

\[ \text{ringing} \]

**FT**

\[ \text{under rect, narrower sinc} \]

**multiply**

acquisition window (truncates high spatial)

**Spatial freq. data**
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \sqrt{I(x)} e^{-i 2\pi k_x x} \]  
Signal eq. \rightarrow \text{fwd problem}

\[ I(x) = \int_{k_x} S(k_x) e^{i 2\pi k_x x} dk_x \]  
Recon eq. \rightarrow \text{inv. problem}

\[ \hat{s} = F \hat{i} \]

\[ \begin{bmatrix} \hat{s} \\ \hat{i} \end{bmatrix} = \sqrt{\begin{bmatrix} F \\ i \end{bmatrix} } \]

Linear "forward solution" matrix vectors have complex entries can build in any measurable priors

\[ F_{x,y,z} = g(x,y) e^{-i\phi(x,y)} e^{-\left( \frac{mT \pm \nu T + \nu T_i}{2} \right)} \]

cal gain at this location
coil phase
T2 decay
B0 error
Freq + phase

multi-coil

\[ \begin{bmatrix} \hat{s} \\ \hat{i} \end{bmatrix} = \begin{bmatrix} F_{\text{coil 1}} \\ F_{\text{coil 2}} \end{bmatrix} \]

naturally incorporates undistorted field map different sensitivity function for each coil contains additional info about some loc. But, need reference scan, lo-res OK (need phase corrections for each coil)

\[ \hat{i} = F^+ \hat{s} \]  
over-determined

More Proner inverse

\[ F^+ = (F^T F)^{-1} F^T \]

\[ (x,y)^2 \rightarrow \text{"small"} \]

\[ F^+ = F^T (F F^T)^{-1} \]

\[ (x,y, \text{coil})^2 \rightarrow 16 \times \text{bigger for 4 coils} \]

\[ \hat{i} = \left[ (F^T)^{-1} F^T \right] \hat{s} \]  
Slice-by-slice assume slice select swamps B0
**FAST SPIN ECHO (FSE)**

- one 90° pulse followed by multiple 180° pulses (e.g., 8)
  - each with a different phase-encode gradient

- each phase "winder" is "unwound" because leftover phase would be re-focused away by 180° (e.g., EPI where it persists between blips)

- the "effective TE" is the TE when center of k-space is collected (largest effect on contrast, largest echo)

- each subsequent echo has more T2 decay: $E_n = e^{-nTE/T2}$

- by arranging to collect $k_y = 0$ early, PD-weighted instead of T2-weighted

- possible to correct different T2-weighting of echoes by estimating T2 curve from $G_y = 0$ echo train

- 3D FSE — like 2D except
  - wind/unwind added to thick slice select (with markers on 180°)

N.B. only one read-rephase, subsequent 180°'s reset from right to left
MULTI-SLAB 3DFSE, PROBLEMS

- echoes die out quickly by $e^{-t/\tau}$
- since echoes after 90° limited to <30, can't fill 3-D k-space in a reasonable time
- SAR constraint $\text{SAR} \propto B_0^2 \theta^3 \Delta B$
  $\Rightarrow 180°$ pulses deposit 4-6x power of $90°$

- multi-slab" is halfway between slices and single-slab

- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses $\Rightarrow$ longer than non-selective 'hard' pulses

- limits speed of covering k-space
SINGLE-SLAB 3D FSE

- Regular FSE (180° pulse train)
- Sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
  - This "storage" in Z-axis preserves magnetization for longer time
  - Smaller flip angles allow much longer echo trains
    - Enough to collect whole plane of 3-D k-space
  - Different than hyper echoes (not symmetric)
  - Contrast must consider STE

Single-slab 3D FSE pulse seq.

SE = \sin(\alpha) \sin^2(\alpha_2 \alpha_2 e^{-2\gamma/12})
STE = \frac{1}{2} \sin(\alpha) \sin(\alpha) \sin(\alpha_3 e^{-4\gamma/12}) e^{-2\gamma/12}

NB: Time to cover k-space is \approx 5X apparent contrast time by/6 of "storage"
(e.g. TE_eff = 585 ms looks like FSE TE = 140 ms)
**FAST GRADIENT ECHO** (GRASS, FLASH, MPRAGE)

- Small tip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- 'Leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

**STEADY-STATE COHERENT** (GRASS, FISP)

- Unwind phase from phase-encode Mx before next pulse (here because TR < TE)
- Unwind read gradient, too

\[ S = k \sin x \left( 1 + \cos x + (1 - \cos x) \frac{1}{T_1/T_2} \right) e^{-T_2/T_1} \]

- T2/T1-weighted contrast (bright CSF)
- Brain 0.7
- Fat 0.3
- CSF 0.7

**STEADY-STATE SPOILED** (SPGR, FLASH)

- Spoil with random gradient (but this still allows some T1 refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

**NON-STEADY STATE, MAGNETIZATION-PREP**

- Preparation pulse, strong T1-weighting
- Contrast varies in spatial-frequency-dependent way

**MP-RAGE**

- Longitudinal mag. not affect much by low angle pulses
- TM
- GM

- Record k_y = 0 here

---

**DIAGRAMS:**

- RF
- G
- Gy
- Gx
- Signal
- TR

- Nominal inversion time
- Effective TI actually time to Gz that records signal

- TR ~ 10 msec
QUANTITATIVE T1 — INTRO, METHODS

Motivation

- Image values are arbitrary/relative (cliff seg’s, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local B1 field can cause contrast variation
  - at 3T, B1 can vary by 25% across the brain
  - this can invert contrast in a fast gradient echo

Pre-scan normalise

- Collect low-res GE image, receive w/ body coil (no coil fall-off)
- Set PARMS to get low GM/WM contrast
- Collect data scan (e.g. MPRAGE) w/surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

T1 divided by T2

- MPRAGE → strong T1-Contrast
- SPACE → T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Problems: distortion different in GE (MPRAGE) and SE (SPACE)
- Noise in regions of low signal

MP2RAGE

1st volume → PD-weighted
2nd copy of volume → more T1-weighted

RF

- N.B. SSFP-like in partition, phase-encode directions
- Convert to -0.5 to 0.5 image: \( S = \text{real} \left( \frac{\tilde{S}_{T1} \cdot \tilde{S}_{T2}}{||\tilde{S}_{T1}||^2 + ||\tilde{S}_{T2}||^2} \right) \)
- Calc. PD & T2 from above (cf. 2 flip angles)
**QUANTITATIVE T1 - HELMS 2-FIIP ANGLE METHOD**

- Start w/ gradient echo signal $s_2$, dropping $T_2$-decay $e^{-TR/T_2}$

\[
S = A \cdot \sin \alpha \cdot \frac{1 - e^{-TR/T_1}}{1 - \cos \alpha \cdot e^{-TR/T_1}}
\]

- Simplicity: Linear Re/estimate
  - $TR < T_1$
  - Linear approx. of exponentials
  - Taylor expansion simplification of $\sin$, $\cos$, drop small terms

- Solve for $T_1$ and $A$ (proton density) given $s_1$, $s_2$ from 2 diff flip angles

\[
T_1 = \frac{2TR}{S_2 - S_1} \quad A = \frac{S_2 - S_1}{S_2 - S_1}
\]

- Problem: Flip angle varies a lot at 3T (e.g., 25%) from nominal requested (e.g., flip series)

**3D Map**

- $a_0$: Spin-echo and stimulated echo (EVI)

- $S_2$ = $k \cdot \sin^2 \alpha \cdot e^{-TE/T_2}$

- $S_{SE} = \frac{k_2 \cdot \sin^2 \alpha \cdot \sin 2\alpha \cdot e^{-TR/T_1}}{S_{SE}}$

- $\alpha = \cos^{-1} \left( \frac{S_{SE} \cdot e^{-TM/T_1}}{S_{STE}} \right)$

- $T_2^*$ est: Add EPI-like echo train to each FLASH excit.

---

**Articles cited:**

- Jiru & Klise (2006)
- Helms et al. (2008)
Echo Planar Imaging (EPI) (another fast gradient echo)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the Gy "blips."

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
**Spin Echo EPI**

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing.

- The excess of oxygen (probably the result of the need to drive O$_2$ into tissue, which requires more O$_2$ in the blood than is actually used) leads to the positive BOLD effect.

- Spin echo corrects (cancels) static $T_2^*$ ($T_2$) dephasing, incl. deoxy.

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing.

- Diffusion exposes spins to different fields (reducing gradient echo dephasing).

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels.

- For $TE \approx 100$ ms, spins diffuse 10's of µm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time.

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion is less likely to expose spin to different fields here).

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells.

- Measure intra/extra w/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels.
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.
- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.
- "Asymmetric spin-echo EPI" arranges for the spin echo to occur T msec before the gradient echo, which gives more T2*-weighting (for ky = 0 echo).

- The 180° pulse rephasing reduces the T2* signal, which is why the partially phased asymmetric spin echo has been more commonly used.
- At higher fields, spin echo EPI is more promising:
  - Signal to noise is higher so we can take spin echo hits.
  - Contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording.
- **COIL FALL-OFF/UNDERSAMPLE/GRAPPA/SENSE**

  - Coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs.
  
  > but what does this look like in k-space?

  - Slow variation in RF field fall-off (e.g. 1-4 cyc/FOV) causes a blur in acquired data in k-space

  > (N.B. not addition!)

  - To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space - at all spatial frequencies!!

  - Simple example w/ "brain" consisting of one spatial freq.

    - Image domain
      
      - Image ("brain")
      
      - Coil fall-off function
      
    - FT
      
      - FT
    

  - N.B. inverse FT of k-space data "smeared" in spat freq.

  > Space is sharp image w/ fall-off (not blurred)

  - "Smeared" means normally orthogonal spatial freq.'s leak to adj. freqs.

  - GRAPPA - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center

  - SENSE - general linear inverse approach

  - N.B.: neither would work unless normally orthogonal spatial freqs. blurred!
- excite multiple slices at once
- function of $G_z$ blips is to shift slices in $G_y$ direction

- this occurs because for given slice, a phase pedestal is added to the entire slice
  - this "Fourier Shift Theorem"
  - [N.B.: different than $B_0$ defect-induced incremented phase errors]

- problem w/ all up $G_z$ blips $\Rightarrow$ phase error builds up

\[ \text{trick#1} \]
- start w/ 2 slices, one at $z=0$, other above
  - if $\pi$ (180°) phase shift used, blip up/down same! (no effect at $z=0$)
  - i.e., move top or bottom replica

\[ \text{trick#2} \]
- for multiple slices not all at $z=0$, phase no longer same for even/odd
  - but can add phase to equilibrate to $k$-space before recon.

\[ \text{trick#3} \]
- for more than 2 slices:
  - 1st
  - even
  - odd
  - 1st
  - even
  - odd
  - etc
MULTI BAND/BLIPPED CHIP (cont.)

- relation between leave-one-out aliasing and nominally fully-sampled SMS

- four alternate lines out wraps image
- SENSE/GRAPPA to fix block coil view swears K-space data
- nominally, w/ SMS we record every line of K-space
- but equivalent to leave alternate out b/c our multi-slice FOV was not big enough

- slice-GRAPPA
  - reg GRAPPA → recon missing lines
  - slice GRAPPA → recon multiple K-spaces i.e. not for each overlapped slices by training on fully-sampled data at beginning of scan

- interslice "leakage block"
  - when training GRAPPA kernel on fully-sampled data, also minimize interslice leakage (split-slice-GRAPPA)
  - can also do regular GRAPPA on top of this reason: for diffusion, loss in S/N from undersample cancelled by shorter TE readout
  - gain from reduced image distortion from shorter readout
- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- entire k-space must be filled before 3D image is reconstructed

- main issue is movement artifact since data assembled from many shots over several secs

- breathing-induced B0 problems in different partitions may cause blur

- since entire volume is excited each shot, potentially higher S/N

- must use smaller flip angle to avoid killing M_L since entire volume excited every partition (e.g. every 80 msec)
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant $\Delta k_x$

- sinusoids in both $G_x$ and $G_y$ give spiral $k$-space trajectory

- constant angular velocity goes too fast at large $k_x, k_y$

- constant linear velocity better but impractical near $k_x = 0, k_y = 0$

- compromise: start constant angular, end constant linear

Constant angular velocity

$w(t) = w_0 T_e$

$k(t) = A t e^{i w_0 t}$

$G(t) = \frac{1}{2} \frac{d}{dt} k(t)$

$= A e^{i w_0 t} + i A w_0 e^{i w_0 t}$

$G_x(t) = A \cos w_0 t - A t w_0 \sin w_0 t$

$G_y(t) = A \sin w_0 t + A t w_0 \cos w_0 t$

Constant linear velocity

$w(t) = w_0 T_e$

$k(t) = A T_e e^{i w_0 T_e}$

$G(t) = \frac{1}{2} \frac{d}{dt} k(t)$

$= \frac{A}{2 T_e} e^{i w_0 T_e} + \frac{A}{2} w_0 e^{i w_0 T_e}$

$G_x(t) = \frac{A}{2 T_e} \cos w_0 T_e + \frac{A}{2} w_0 \cos w_0 T_e$

$G_y(t) = \frac{A}{2 T_e} \sin w_0 T_e + \frac{A}{2} w_0 \sin w_0 T_e$
**SPIRAL 3D IR FSE** (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)

Possible to present sign
- High, uniform contrast, but lots of waiting (TI), high BW

RF

\[ 180° \text{(prep)} \rightarrow 90° \rightarrow 180° \rightarrow 180° \rightarrow 180° \] (prep)

\[ G_z \]

\[ G_y \]

\[ G_x \]

\[ \text{Sig.} \]

3D k-space ("stack of spirals")

Loop order

Spiral interleaves
- k_x interleaves
- k_x echoes
- k_z echoes (after one 90°)
**Phase Errors & Echo-Centering Errors**

Anything that causes a deviation of the $B_0$ field strength from the expected value $(B_0, z + G_{x,z} x + G_{y,z} y + G_{z,z} z)$ changes precision frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

### Fourier Shift Theorem

Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x-x_0) = \int_{k_x} e^{-i2\pi k_{0,x} x} S(k_x) e^{i2\pi k_{0,x} x} dk_x$$

$x_0$ - offset in image

N.B.: This is a "pedestal" of phase, not a gradient.

### Echo Centering Error

- If realignment of all spins ($k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.

- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction; magnitude image unchanged.

### Fourier freq. shift theorem

Frequency shift in freq. domain causes phase shift in spatial.

$$e^{i2\pi k_{x0} x} I(x) = \int_{k_x} S(k_x-k_{x0}) e^{i2\pi k_{x0} x} dk_x$$

$S(k_x-k_{x0})$ - offset in spatial freq. space
**FAST SCAN ARTIFACTS**

**EPI vs. Spiral**

**EPI**
- $G_x$ readout gradient stronger → field defects smaller percentage
  - less deformation of $k_x$ (vertical stripe components)
- $G_y$ "blips" are weak and total $G_y$ record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the $x$-direction, for example, maps and unmaps phase as a function of $x$-position
- but $G_x$ big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates

The lack of blurring has lead to a preference for EPI, despite the substantial image shifts.

**Spiral**
- with center-out spirals phase errors accumulate in a radial direction
- thus, higher spatial frequencies have more error (= more shearing)

- for spurious $x$-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq.

- For defects with more complex contains in the $y$-direction (than linear, as above), the vertical shifts (in EPI) will vary with $y$-position and may result in signals from different $y$-positions being reconstructed on top of each other.
**IMAGE-SPACE VIEW OF LOCALIZED BØ DEFECT, EFFECT ON RECON**

- **Localized BØ defects often arise from air pockets embedded in tissue**
  - Air in middle/outer ear → indentation in inferior Temporal lobe
  - An under defactory epithelium → orbitofrontal cty, ant, thal. compression

**Collect one data (k-space) point**

\[
\begin{align*}
\text{4 cycles of phase in y-dir (z-position)} & \quad + \\
\text{Localized BØ defect} & \quad \rightarrow \\
\text{Complex multiply} & \quad (= \text{correlate sin, cos with brain}) \\
\text{Brain structure} & \quad \text{sampled with} \\
\text{distorted stripes} & \quad \text{one complex number}
\end{align*}
\]

**Reconstruction from distorted data points**

\[
\begin{align*}
\ldots & \quad + \\
\text{[Undistorted stripes used by inverse FFT]} & \quad \text{x} \\
\text{Amplitude and phase} \quad \text{of this component} & \quad + \\
\text{[Same for 5 cycles]} & \quad \ldots \\
\end{align*}
\]

**Local upward displacement image phase (phase encode dir)**

**N.B.:** Image shift only occurs if shift spatiotemporal sampled w/ successively later echos (see next page)

**Close-up of distorted phase stripes (one cycle)**

- **Spatial information can be lost when continuous changes in phase are flattened by BØ defect**

\[
\begin{align*}
\text{Spatial information can be lost when continuous changes} & \quad \text{in phase are flattened by BØ defect} \\
\phi & \quad \text{defect}
\end{align*}
\]

- **Shifts can pile multiple pixels on top of each other into one bright pixel**

- **Local estimates of BØ needed to correct images**
  1. **Fieldmap method:** multiple TE's to est. local BØ from \(\frac{\Delta T}{\Delta E}\) slope
  2. **Point-spread-function:** extra phase encode to estimate PSF (should be \(S\)-function) deconvolve distorted image in phase-encode direction
LOCALIZED B₀ DEFECT, EFFECT ON RECON

- when local B₀ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive k̇y line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- however, with w/EPI, static B₀ defect causes more and more local displacement of image phase stripes for each additional k̇y line

- that is, later lines have greater spat. freq. offset
  - effectively stretches k-space in ky direction
  - same num. samples to higher spatial freq.
    - shrinks FOV (squishes voxels—see FOV page)

- when image is reconstructed, region with local B₀ defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:
  1) static local ΔB₀ defect
  2) successive increases in phase error for successive spat. freq. measurements during long EPI readout
  3) small size of ky phase encode blips relative to B₀ defect (much less of this effect in freq. encode direction)

- respiration (which affect B₀) in 3D FLASH might cause similar effect within k̄₂ partition (if successive spat. freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to imprint a linear variation onto the $z$-component of the $B$ field $- B_z -$ in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (esp. printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion:
  - A non-linear slice-select gradient will excite a curved slice.
  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently:
  - For 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!)

- This can result in errors approaching 1 cm in functional overlays.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

- These effects do not build up over time in phase-encode direction since they only occur when gradients are turned on.

- Fourier shift theorem: these distortions are predictable and can be corrected.

- That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
SHIMMING AND $B_0$-MAPPING

- Passive iron shims inserted to flatten $B_0$ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $B_0$ field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc.) (= several hundred ppm)

Linear shim coils impose gradients in $x$, $y$, and $z$
Higher order shims impose higher order spherical harmonic field components (e.g. $z^2$)

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $B_0$ field
- Local resonance offsets caused by $B_0$ defects estimated from images
  \[ \Rightarrow \text{e.g., sample phase at multiple echo times} \]

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  \[ \Rightarrow \text{this only corrects spatially gradual field defects} \]
  \[ \Rightarrow \text{local defects due to air in sinuses much higher order than shims} \]

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to un warp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq. encode gradient is strong relative to defects)
**NAVIGATOR ECHOES**

- **1D navigator**
  - **Bϕ drift problem**
    - slow up/down drifts in Bϕ continuously occur
    - a pedestal in Bϕ is pedestal in phase (not gradient)
    - which causes spatial shift (Fourier shift theorem)
    - in EPI, mainly affects phase-encode dir b/c small slip
      in readout dir
    - result is successive volumes drift in phase encode dir

**Gradient balance problem**

- unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
  causing N/2 (Nyquist) ghosting ➔ another phase error

---

**3D navigator**: collect 3D sphere in k-space

- rotation of object ➔ rotation of k-space amplitude pattern
- translation of object ➔ phase shift of k-space phase (Fourier shift)
- sample at sufficiently small radius to pick up high spatial freq features
- N.B.: excite whole volume
- do N,S hemispheres separately (less T2**: cancel EPI-like error accumulation**)

Walch et al. (2002) MRM

\[ z(n) = \frac{2n - N - 1}{N} \]

\[ y(n) = \cos\left(\frac{N\pi}{2} \sin^{-1}(z(n))\right) \tanh(1 - z^2(n)) \]

\[ x(n) = \sin\left(\frac{N\pi}{2} \sin^{-1}(z(n))\right) \tanh(1 - z^2(n)) \]

(skip poles — slew rate too high)

- can be used for prospective motion correction (rotate, translate w/ gradients)
- better estimate, because of speed, than full TR & EPI images (27 ms vs. 2.4 sec)
- may need to smooth rot,trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES

- **Receive coil inhomogeneities** alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way. Variations can be used (e.g., GRAPPA, SENSE) and/or corrected.

- **Transmit coil inhomogeneities** affect the flip angle in a spatially varying way (can affect contrast: FLASH). Potentially worse (why local transmit is still in progress). Usual fix by using a large transmit coil (e.g., body coil).

- RF penetration at higher fields is less uniform:
  1) Decreased RF wavelength (closer to size of head) at higher freq.
  2) Increased permittivity (ε) and conductivity (σ) at higher field.

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain).

- Different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP normalization) (“pre-scan normalize”)

  - Record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)

  - Divide small coil/ body coil at each voxel to determine receive field

  - Use receive field to normalize main image(s)

  [See also: qT1, MP2RAGE, T1/T2]
DIFFUSION - WEIGHTED IMAGING

Simple diffusion weighting

- Spins acquire phase during first \( T_1 \)
- If spins diffuse (move) along gradient by time \( T \), signal is lost because negative \( T_2 \) doesn't re-phase
- Attenuation: 
  \[
  A(D) = \frac{S_0}{S} = e^{-bD}
  \]
  where 
  \( b = y^2 G^2 T^2 (T - 8T/3) \)

- "Apparent diffusion coefficient" \( \bar{D} \)
- After obtaining 3x3 tensor, calc. eigenvectors/values to find orientation & shape of diffusion ellipsoid
  - Two useful scalar values from 3 eigenvalues:
    - Apparent diffusion coefficient
    - Mean diffusivity: 
      \[
      ADX = MD = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}
      \]

1) Anisotropic Diffusion (Gaussian)

- Measure \( D \) along multiple axes
- Have to measure tensor, not scalar
- Even for determining one primary direction
  - isotropic: diagonal, need minimum of 6 diff. measurement directions
- Non-Gaussian

2) Length Scale by multiple \( b \)-values

- fit line to semi-log signal as funct of \( b \)
- If not straight line: multi-exponential, e.g.
  - hi ADC/fast/extravasation vs low ADC/slow/intracellular
  - vs time or distance for tissue diffusion

- long \( T \), gradient can give spurious T2-weighting
- can use stimulated echo to get long T w/o less T2-weighting

- 90° RF - ST - 180° RF - 90° RF

- Without 3rd number (0)
  - \( x \) & \( y \) projections same

- Tract Tracing
  - 1) Markov process
  - 2) crossing fibers
  - 3) "freeway ramp" prob
  - 4) Sharp turns into gyri

- Voxels

- Diffusion surface (non-Gaussian)

- Need to measure diffusion in many directions (>6) to properly characterize even 2 main directions

- Scalar diffusion
- Diffusion tensor measurement direction
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- Spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)
  - Allows longer TE
  - Flips M_z so rephase gradient: same sign as de-phase

- Eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion

- "Doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/partic. time constants)
  - Also, keep crushers orthogonal to diffusion-encoding gradients

Nagy et al. (2014) MRM

\[ y_{TRSE} = 0 = y_1 - y_2 - y_3 + y_4 \]
Perfusion - Arterial Spin Labeling

Basic idea:
- tag blood below area of interest
- collect control & tagged images
- assume directional input flow
- tag is 180° pulse
- sign not problem when delay long enough (see below)
- continuous ASL (CASL) — [continuously tag a plane, greatest on, blood gets adiabatically inverted as it passes through location w/ coronary resonance tag]
- pseudo-contin. ASL (pCASL) — see next
- pulsed ASL (PASL) — e.g., Epistar, FAIR, PICORE, QUIPPS II
  - tag block of tissue below slice(s)
  - small delay between control and tag (~10s)
  - requires accurate balancing of control & tag images, control mag. transfer
  - transit delays — biggest confounding factor
  - contrast problems:
    - transit delays
    - venous clearance (vs. microvessels, which get stuck!)
  - solutions for quantitative
    - in order delay so all spins arrive into low-velocity capillaries
    - kill end of tag to reduce spatial variation of tag

QUIPPS II — quantitative perfusion

1) pre-saturate spins in target slices
2) tag - 180° pulse below slices
3) control - 180° pulse above slices (to control off-resonance)
4) Saturate tagged block to end tag (TI)

ΔM = flow × [2M₀ TI, e⁻ᵀ₁₂/ᵀ₁A]

Turbo ASL
- use TI longer than TR
- omit QUIPPS tag ending
- tag pulse
- tag image
- control image
- control pulse
- control mag.

ΔM ≈ flow × [2M₀ TI, e⁻ᵀ₁₂/ᵀ₁A]

Alternate tag and control, GRE TE = 30 ms
- control-tag → flow
- control+tag → BOLD
- tag-control-tag-control...

Dual echo spiral
- k=0 TE early =⇒ high S/N flow
- TE = 30 ms =⇒ BOLD

Con extract flow and BOLD adjacent substrates minimize movement artifact

[Image of a diagram showing the perfusion process with tags and control images]
PERFUSION - pCASL

- Original CASL (continuous arterial spin labeling) requires:
  - RF on continuously to adiabatically invert blooded flowing
  through one plane
  - can only image one slice (bc dephasing from gradient)
  - hard to keep RF on continuously on modern scanner (esp. BC)
  - can use special purpose RF transmit (separate xmt channel)

A) original CASL

\[
\begin{align*}
\text{RF} & \quad \text{image formation module ("readout")} \\
\text{G}_z & \quad \text{[multiple possibilities]}
\end{align*}
\]

(on pulsed!)

- Problem: multiple pulser create aliased slice planes

\[
\text{RF}(t) = \frac{1}{\Delta t} \text{comb} \left( \frac{t}{\Delta t} \right) \ast \text{rect} \left( \frac{t}{\delta} \right)
\]

[use: convolution of 2 funct equals multiplying their FTs]

\[
F[\text{RF}(t)] = \text{comb} \left( 6 \Delta t \right) \ast \delta \text{sinc} \left( \pi \delta \lambda \right)
\]

- aliased labeling planes at: \( \lambda = n/\Delta t \) in frequency space, modulated by broad sinc()
- Use Hannings or hyperbolic secant to reduce replicas

B) pCASL - pseudo continuous arterial spin labeling  
Dai, Alsop (2008)

\[
\begin{align*}
\text{RF} & \quad \text{image formation module ("readout")} \\
\text{G}_z & \quad \text{begin readout}
\end{align*}
\]

\[
\text{comb}(t) = \sum_{n} \delta(t-n)
\]

\[
\text{rect}(t) = \begin{cases} 
1 & \text{if } |t| < \frac{\delta}{2} \\
0 & \text{otherwise}
\end{cases}
\]

- for const \( G_z \):

\[
z = \frac{n}{Y G_z \Delta t}
\]

- i.e. spacing inversely proportional to \( \Delta t \)

C) pCASL w/ shaped gradients

- Tag pulses have phase offset respecting gradient
- Control identical except every other has +\( \pi \) phase
- No net flip
**OFF RESONANCE EXCITATION**

- **Main Idea:** Examine evolution of $\vec{M}$ vector in rotating coord syst set to "off-resonance" $\vec{B}_1$ field freq ($\omega_{rf}$), not Larmor freq of $\vec{M}$ ($\omega_0$).

- Normally, if rotating coord freq set to Larmor freq ($\omega_{rf} = \omega_0$), an actually precessing $\vec{M}$ will be stationary (ignoring decay) → implies effective $B_z = 0$ in rotating.

- Now, move $\vec{M}$ to rotating coord syst at $\vec{B}_1$ freq lower than $\omega_0$ (assume $\vec{B}_1 = 0$ = off): existing $\vec{M}$ will now appear to precess around $z$-axis:

**N.B.:** This is precession in already rotating coordinate system. (slow relative to $\omega_0$)

$$\Delta \omega_0 = \omega_0 - \omega_{rf}$$

freq of precession in rotating coordinate syst

Larmor rotation freq $\vec{B}_1$

(incorrectly set rotating coord syst freq)

- Thus, viewing $\vec{M}$ vector in off-resonance rotating coord syst makes it look like additional $B_z$ field is causing extra precession.

- "Extra" $B_z$ component is proportional to $\Delta \omega_0$: offset

  $\Rightarrow$ can be pos or neg: rota coord too low → pos $B_z$
  rota freq too high → neg $B_z$

- Extra $B_z$ adds to $\vec{B}_1$ resulting in slow precession around tipped axis: $\vec{B}_{eff}$ (effective)

- Extra $B_z$ from any gradient → same effect on $\Delta \omega_0$

  (changes $\omega_0$ instead of changing $\omega_{rf}$)

$$\vec{B}_{eff} = \left(\frac{\Delta \omega_0}{Y}\right) \hat{z} + B_{z}\hat{i} + \vec{B}_1 \hat{i}$$

  effective $\vec{B}$ in rotating frame set to $\vec{B}_1$ freq

- Extra $B_z$ from Larmor-$\vec{B}_1$ freq mismatch (pos or neg)

  (if on-res. → 0)

- Transverse RF stim (additional source of $\vec{B}_{eff}$ tilt)

- RF: sweep freq $\omega_0$: constant $\omega_0$, $\omega_0$: sweeps because spins flow along gradient direction

- CASL tag RF: drive freq
Spectroscopy + Image

- Chemical shift: small displacement resonant freq due to shielding of target nucleus (e.g., H) by surrounding electron orbitals.

  - E.g., acetic acid:

    - Oxygen attracts electron so less shielding of target nucleus.
    - 3 of these H's (more shielded).
    - 1 of these H's (less shielded).

- How we get chemical shift spectrum:

  - Data before FT is a series of time-domain samples of the mix of shifted freq offsets.
  - FT turns data into "shiftspectrum".

  - N.B.: opposite "direction" of FTs!

### NMR

<table>
<thead>
<tr>
<th>Signal</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time-domain FT spectrum</td>
<td>Shift freq. samples</td>
</tr>
<tr>
<td>Oscillation (shift)</td>
<td>Spatial freq. samples</td>
</tr>
<tr>
<td>Samples</td>
<td>Spatial object (like time-domain signal)</td>
</tr>
</tbody>
</table>

### MRI

Pulse Sequence

- Since we are already using phase (freq.) encoding for space, we need an "extra dimension" w/ all gradients OFF!
- Use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal $\text{and FT-it like chemists do!}$
PRESS, MEGA-PRESS

 usu. single voxel by using 3 orthog. slice selects
 (10 can add PE gradients & more excitations to get multiple vox)

PRESS — 3 orthog. slice select

MEGA-PRESS — add "editing" RFs to suppress solvent (water)

It's OK, now to think of multiple freqs here.

FT to get shift spectrum

G₁, G₃ — asymmetric spoilers to dephase spins in bandwidth of selective MEGA pulses

G₂ — G₁ symmetric around 2½ 180°
**Phase-encoded Stimulus & Analysis**

Map polar angle

Map frequency

Map eccentricity

Map prox/distal axis, Road maps

Periodic stimuli (phase-encoded) e.g. 8 cycles at 64 sec/cycle

Typically 0.5-5% amplitude

Strongly periodically activated single voxel time course

Remove constant (avg) and linear trend

Real

Imaginary

FFT, convert to A, \( \phi \)

Reversed CCW vector average CW significance

CCW significance (complex)

Calculate significance
- Ratio between amplitude at stimulus frequency (=signal) and average of amplitudes at other frequencies (=noise)
- Ignore harmonics, lo freq (=movement)

Smooth
- Vector average of complex significance \((A, \phi)\) with that at nearest neighbor surface points

Display
- Plot phase using hue and saturation to indicate significance

Delay correction
- Record responses to opposite directions of stimulus (ccw/lcw, in/out, up/down)
- Vector average after reversing angle of one
  - Penalizes inconsistent more than just avg of angles
**Convolution**

\[ h(x) = f(x) \ast g(x) = \int_{-\infty}^{+\infty} f(z) \cdot g(x-z) \, dz \]

- **Definition of Convolution** \( f \ast g(x) \)
- **Commutative**

**Why reverse makes sense**

Impulse response function (HDR)

**Intuitive non-reversed view of convolution output**

**NB.** Cross-correlation same as convolution except no reversed \( g(x+z) \) instead of \( g(x-z) \)

**NB.** Auto-correlation same, except no reversed and use same function for both \( f, g \)

How to calculate convolution output for this time point (only 3 terms in sum, all others zero)
**General Linear Model (GLM)**

\[
\hat{y} = Xh + \hat{\beta}b + \hat{n}
\]

Data = design \cdot HDR + drift weights + noise

**Goal:** solve for the hemodynamic response functions, \( \hat{h} \)

**Block design:** better to detect resps.
**Event-related design:** better to measure HDR shape

\[
y_{\text{predicted}} = X\hat{\beta} + \hat{n}
\]

- **Simpler: Preconvolve**
  1. conv. \( X \) if fixed \( \hat{h} \)
  2. solve for \( \hat{\beta} \)
        \( e.g. \ y = X\hat{\beta} + \hat{n} \)

**Maximum Likelihood Estimate**

1) Assume white noise, solve for \( \hat{h} \)

\[
\hat{h} = (X^T P_s^+ X)^{-1} X^T P_s^+ y
\]

where \( P_s^+ = I - S (S^T S)^{-1} S^T \)

\[
\leftarrow \text{projection matrix that removes part of vector that lies in } S \text{ space}
\]

2) \( \hat{h} = (X_\perp^T X_\perp)^{-1} X_\perp^T y \) where \( X_\perp = P_s^X \)

3) **Significance** (how to constrct F-ratio)

\[
F = \frac{N-k-l}{k} \left[ \frac{y^T (P_{ks} - P_s) y}{y^T (I - P_{ks}) y} \right]
\]

\( P_{ks} \) = projects data on essential subspace
\( P_s \) = projects data onto nuisance subspace

**Orthogonal cols:** most efficient to minimize trace \([X_i X_i]^T\) to get

**Basic vectors of Signal Space**

**Basic vectors of Error Space**

\[
\text{N.B. a point in these spaces is a timecourse}
\]

**Matrix notation for discrete convolution of stimulus pattern with hemodynamic resps.**

**Stat-3**
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- With no nuisance functions (S), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

\[
\hat{y} = X\hat{h} + \hat{e}
\]

\[
\tilde{y} = \hat{X}\hat{h}
\]

Projection matrix, \( \hat{P_x} \), operates on \( \tilde{y} \) to give projection of data into experiment space, \( X \).

- When nuisance functions, \( S \), are considered, problem: \( S \) may not be orthogonal to \( X \).

\[
X = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\]

For example: linear trend not orthogonal to std. block design.

\[
\text{block design}
\]

\[
\text{linear trend}
\]

\[
\text{calc. dot prod}
\]

\[
\text{sum: not 0}
\]

**Geometric Picture**

(Liu et al., 2001, Neuroimage)

\( X_S \): Space of data modeled by all exp. design and nuisance.

\( E_y \): Error (e) not explained by exp. design and nuisance (F denom.)

\( P_y \): Orthogonal projection onto nuisance (\( P_y \)).

\( E_y \): Orthogonal projection onto experimental design (\( E_y \)).

\( P_{xy} \): Orthogonal projection onto experimental design and nuisance.

\( \text{data} \): Data explained by reference.

\( \text{unit } S \): How much more of data you can explain by adding experimental design (F numer.)

\( \text{unit } X \): Same as projection onto reference only in special case where \( S \perp X \).
WHY USE SURFACES?

- raw MRI data is a 2D flat slice or a 3D volume
  \[ I(x, y) \text{ or } I(x, y, z) \]
  but...

1) the neocortex (and cerebellar cortex) are thin, folded 2D sheets
   - cortex starts as smooth "balloon" →
   - major sulci, temporal lobe form →
   - great size increase, "crinkles" form

2) neocortex contains many topological maps along its surface
   - retinotopy
   - tonotopy
   - somatotopy
   - musculotopy
   - plus higher level maps → \( \approx \frac{2}{3} \) of its area

3) surface displays allow seeing (almost) all of data at once
   - only 30% exposed → everything visible

4) differences in major sulci make 3D-based alignment difficult
   - e.g. STS, monkey-like IPS vs. postcentral
   - requires mapping to
   - extremely anisotropic def.

5) idiosyncratic sulcal crinkles
   - these introduce additional noise into alignment in 3D
   - exact position of crinkles unlikely to have functional implications (the 3D align might respect them)
1) MNI auto-Talairach \( \rightarrow \) generates 4x4 matrix
- make average brain target (blurry)
- blur target (Further), blur single brain (a lot), gradient descent on \( x \times x \)
- repeat w/ less blurring of avg target and current brain
- problems: variable neck cut off; only 2 points near center of brain!
  \( \Rightarrow \) but much better than standard! \( \leq \) fit to bounding box

2) Intensity Normalization (output: "T1")
- histogram of pixel values in 10 mm thick HTR slices
- smooth histogram
- peak find to get initial estimate of white matter
- discard outlier peaks across slices
- fit splines to peaks across slices
  \( \Rightarrow \) interpolated scaling factor \( \rightarrow \) to HTR
- scale each pixel so WM peak is 1.0
- refine estimate to interpolate in 3D
  \( \Rightarrow \) find points in \( 5 \times 5 \times 5 \) within 10% of WM, get new scale for them
  \( \Rightarrow \) build Voronoi to interpolate scales (6 iterations)
  \( \Rightarrow \) re-scale each voxel

3) Skull Stripping (output: "brain")
- "shrink-wrap" algorithm
- start with ellipsoidal template
- minimize brain penetration and curvature
  \( \Rightarrow \) curvature: spring force (from center-to-neighbor vector sum)
  \( \Rightarrow \) brain penetration
    apply force along surface normal that prevents surface from entering gray matter
    \( \Rightarrow \) decompose into 1 and tangential (local normal from summed normal cross products)
SEGMENTATION & SURFACE RECON

Spring force in detail

- Implementing a "force" is like directly constructing the operator that minimizes something (without first defining the 'something')
- More formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g., (skull strip, original Dale & Sereno surface refinement)

\[ \mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t) \]

rule for each vertex, \( \mathbf{r}_{\text{center}} \)

\[ \mathbf{F}_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (I - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \]

\[ + \lambda_{\text{normal}} \sum_{\text{neigh}} (\mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) - \frac{1}{\#\text{vertices}} \sum_{v} (\mathbf{n}_v \mathbf{n}_v^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_v) \]

\[ \mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \frac{30}{d} \max \left(0, \tanh \left[I \left( \mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}} \right) - I_{\text{thresh}} \right] \right) \]

\[ \text{max force} \]

\[ \text{saturation at} \ 1.0 \]

\[ \text{max force product} = 1.0 \]

\[ \text{outside} \ (\text{dark}) \]

\[ \text{skin} \ (\text{light}) \]

\[ \text{skull} \ (\text{dark-light-dark}) \]

\[ \text{GM} \]

\[ \text{ideal skull strip} \]

\[ \text{WM} \]

\[ \text{etc} \]

Snapshot of surface and "core sample" from one vertex
SEGMENTATION & SURFACE RECON

4) Non-isotropic filtering (output: "win") — "floss" and "speckle"
- preliminary hard thresholds: output
  \[ \text{GM: WM} \rightarrow \text{blood vessels (bright)} \]
  also truncated to black
- find ambiguous/boundary voxels
  \( \rightarrow 20\% \) or more of 26 immediate neighbors different
- find plane of least variance
  for each direction (from icosahedral super-tessellation)
  consider \( 5 \times 5 \times 5 \) volume around 1 voxel
  find plane of least variance in this hemisphere
  medium filter w/ hysteresis
  \( \rightarrow \) if \( 60\% \) of within-slab differ, reverse classification
  \( \rightarrow \) "flosses" sulci without blurring

5) Find cutting planes
- midbrain
- callosum, to separate hemispheres (SAG)
- midbrain, to avoid fill into cerebellum (T1R)
  \( \rightarrow \) Talairach to start:
  fill WM in SAG or T1R till min area

6) Region-growing to define connected parts (output: "filled")
- inside-out, outside-in, inside-out — for each hemisphere
- up/down cycles within each plane
- plane-by-plane
- "wormhole filter" (3x3x3 = center + 26)
  \( \rightarrow \) fill (unfilled) voxel if 66% neighbors differ
- eliminates structures within, 1-D structure
7) **Surface Tessellation** *(output: rh.orig, lh.orig)*

- variable num neighbors possible!
- quads to triangles

- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  # so cross-products oriented properly

- long list of values associated with each numbered vertex
  - position (orig, morphed)
  - area (orig, morphed)
  - curvature (intrinsic, Gaussian)
  - "sulcuscness" (summation of movement during unfolding)
  - cortical thickness
  - fMRI data
  - EEG/MEG dipole strength

- separate fMRI data set must be aligned, sampled

- fMRI voxels larger
  - sample at each surface vertex
  - nearest-neighbor "soap bubble" smoothing to interpolate data onto hi-res mesh

- some quantities only well-defined on surface
  - gradient of magnitude of cortical map measure (e.g., eccentricity)
SEGMENTATION & SURFACE RECON  
Smooth, inflate, final surfaces

- smoothing/inflation/WM, pial done as derivative of energy functional

\[ J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}} \]

- total scalar error to minimize
- scalar tangential error (fixed by redistributing vertices)
  \[ \text{small} \ (0.25) \]  
- scalar curvature error (fixed by reducing curvature)
  \[ \text{smaller} \ (0.075) \]  
- Scalar image error (fixed by moving toward target image value)

\[ J_{\text{normal}} = \frac{1}{2 \#\text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right]^2 \]

- across all vertices, curvature error
  \[ \frac{1}{2} \text{ so no coefficient on derivative} \]
- across all vertices of one vertex
  \[ \mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}} \]
- vertex unit normal
  \[ \text{vector from current center to one neighbor (position vector diff)} \]
- target unit normal
  \[ \text{vector from current center to one neighbor (position vector diff)} \]

\[ J_{\text{tangential}} = \frac{1}{2 \#\text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right]^2 + \left[ \mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right]^2 \]

- "squishing" of mesh
- \( \mathbf{x}\)-direction in tangent plane
- \( \mathbf{y}\)-direction in tangent plane
- \( \mathbf{t}^x_{\text{center}}, \mathbf{t}^y_{\text{center}} \) are first 2 eigenvectors at neighbor vector cloud (\( n \) is third)

\[ J_{\text{image}} = \frac{1}{2 \#\text{vert}} \sum_{\text{centers}} \left[ I_{\text{tag}} - I_{\text{center}} \right]^2 \]

- \( I_{\text{tag}} \) for WM: mean of voxels labeled WM in 5 mm neighborhood
- \( I_{\text{tag}} \) for pia: global - small num for C.S.F.-like

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-interest test

\[ \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{tag}} - I_{\text{center}} \right] \nabla I_{\text{center}} \]

- \( \lambda_{\text{image}} \) gone bc const
- vector = calculate gradient on image (first blur w/ Gaussian)

\[ \sum_{\text{neighbors}} \lambda_{\text{normal}} \left[ \nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right] (\nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})) \]

- x-component of tangential

\[ \sum_{\text{neighbors}} \left[ \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right] (\mathbf{t}^x_{\text{center}}) + \]

- y-component of tangential

\[ \sum_{\text{neighbors}} \left[ \mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right] (\mathbf{t}^y_{\text{center}}) \]

\[ \lambda_{\text{normal}} \left[ \nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right] (\nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})) \]

\[ \lambda_{\text{normal}} \left[ \nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right] (\nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})) \]

\[ \lambda_{\text{normal}} \left[ \nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right] (\nabla_{\text{center}} \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})) \]

N.B.: eq. 9 in Delf, Frich, Sareen different - and incorrect!
**SULCUS-BASED CROSS-SUB. ALIGN**

- Use summed perpendic. vertex move during inflation as vtx measure of "sulcus-ness"
- Add term to error function, $J$: "sulcus-ness" error

$$J_{sulc} = \frac{1}{2\#\text{verts}} \sum_{\text{verts}} (S_{\text{sub}} - S_{\text{target}})_{(\text{vtx})}^2$$

- Find reg of steepest uphill direction of change in sulcus-ness of target

$$\frac{\partial J_{sulc}}{\partial \text{target}} = \lambda_{sulc} [S_{\text{sub}} - S_{\text{target}}]_{(\text{vtx})} (-\sqrt{S_{\text{target}}}_{(\text{vtx})})$$

- Bootstrap
  - Morph to one brain
  - Make avg. target
  - Re-morph to avg. target

- Smooth WM • Inflated • Sphere • Registered Sphere

- Each sub's native surf has diff # vertices

- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)

- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal crinkles (free average "inflated")
  - Inflated: retains orig area, correct sulc/gyus ratio ("inflated-avg")

- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)
**Source of EEG/MEG**

- **PSPs**
  - Anisotropic cables
  - Aligned spatially
  - Coherent/biased stim on one end

- **Head**
  1) Local dipole
  2) EEG through skull, skin
  3) Sweating because skull 1/80 conductivity of brain

- **MEG**
  - Radial dipoles lost
  - Tangential dipole generates Gabor-like scalp distribution of $\vec{B}$ field

**N.B.:** Spikes only detected by 15 μm microelectrode in gray matter!

**Dendritic “cable”**

**N.B.:** $R_M > R_L$

"Closed field" (invisible at distance)
**Intracortical Circuits & Origin of EEG**

**Cell Types**
- **Excitatory (Spiny)**: pyramidal, spiny stellate (e.g., V1, layer 4C)
- **Inhibitory (Smooth)**: basket, double bouquet, chandelier, clutch

**Circuits**
- Huge complexity
- First principal components: input → layer 4 → layer 2/3 → feedforward
  - Micro-electrode recording (e.g., 10 μm tip):
    - High pass → spikes
    - Low pass → local field potentials
- Spikes only recordable in gray matter
- White matter spikes only recordable with pipette with very fine tip b/c inward & outward currents so spatially close in axon/spike (>1 μm)

**Intra-inter cortical connections cartoon**

- "Lower" (e.g., V1):
  - 2/3 feedforward
  - 4 input
  - 5 motor output
  - 6 feedback
  - Ascending input (e.g., dLGN)

- "Higher" (e.g., V2):
  - 2/3
  - 4
  - 5
  - 6
  - Feedback avoids layer 4
  - Motor striatum

- Spike is inverted here
- Axon initial segment

- Opposite polarity here!
GRADIENT, DIVERGENCE, CURL

**Gradient** \(\nabla\) (generalized derivative)

\[
\nabla \mathbf{s}(\mathbf{r}) = \frac{\partial s(\mathbf{r})}{\partial x} \hat{i} + \frac{\partial s(\mathbf{r})}{\partial y} \hat{j} + \frac{\partial s(\mathbf{r})}{\partial z} \hat{k}
\]

- **Divergence** \(\nabla \cdot \mathbf{V}\) (diviv, "dot product")

\[
\nabla \cdot \mathbf{V}(\mathbf{r}) = \frac{\partial v_x(\mathbf{r})}{\partial x} + \frac{\partial v_y(\mathbf{r})}{\partial y} + \frac{\partial v_z(\mathbf{r})}{\partial z}
\]

- **Curl** \(\nabla \times \mathbf{V}\) (diviv, "cross product")

\[
\nabla \times \mathbf{V}(\mathbf{r}) = \left( \frac{\partial v_z(\mathbf{r})}{\partial y} - \frac{\partial v_y(\mathbf{r})}{\partial z} \right) \hat{i} + \left( \frac{\partial v_x(\mathbf{r})}{\partial z} - \frac{\partial v_z(\mathbf{r})}{\partial x} \right) \hat{j} + \left( \frac{\partial v_y(\mathbf{r})}{\partial x} - \frac{\partial v_x(\mathbf{r})}{\partial y} \right) \hat{k}
\]

**Vector identities**

\[
\nabla \times \nabla \mathbf{s} = \mathbf{0}
\]

- **Curl of the gradient of any scalar field is zero**

\[
\nabla \cdot (\nabla \mathbf{A}) = \mathbf{0}
\]

- **Divergence of the curl of any vector field is zero**

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}
\]
Potential (Φ), Electric Field (∇Φ) \& CSD (∇ \cdot (-∇Φ) = \nabla^2 \Phi)

Low-frequency field approximation
- Electric fields uncoupled from magnetic (\(\nu_0\) electromagnetic radiation)
  \(\Rightarrow\) Pre-Maxwellian approx. (EEG freqs \(\ll 1\) MHz)
- Calculate electric fields as if magnetic fields don't exist
- Calculate magnetic fields strictly from distribution of currents
- Ignore capacitive effects, too

Scalar potential, \(\Phi\) (what we measure with electrode)

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \]

\(\vec{E}\) is Laplacian of \(\Phi\) (= div \(\vec{E}\))

\(\nabla \cdot (-\nabla \Phi) = \text{Scalar field} = -\left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] = -\nabla^2 \Phi\)

3D CSD gold standard (rat BAER paper)

\(\Phi\) data \(\xrightarrow{\nabla} \nabla \Phi \xrightarrow{\nabla \cdot (-\nabla \Phi)} \text{Scalar field source/sink movie as function of z}\)
1D CSD
- raw, event-related signal relative to ground, \( \frac{\text{d}I}{\text{d}t} \) (e.g., skull)
- spikes (upside down of extracellular)
- LFP (local field potential)
- both types of data can be recorded from same electrode
- rationale: CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet
- assume approx constant (\( \approx 0 \)) parallel to cortex
- recording sites (e.g., by slowly withdrawing electrode tip)
- high pass
- low pass

2D CSD
- 2D array of electrodes on pial surface or on scalp
- rationale: all electrodes record along same surface so assume depth profiles are constant
- \( \nabla^2 \) means find spatial (i.e., 1D depth) curvature of potential
- discrete approx: center - \( \frac{\text{above} + \text{below}}{2} \)
- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies sink!
- for scalp recordings, sources and sinks are at the scalp
- (not a depth line method unless done in 3D)

\( \nabla^2 \Phi \) (don't confuse dim of calc w/ always a second deriv.)
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukow & Barth, 1998)

-- CSD --

Layer 4

- 50 msec

P1

N1

P2

- phase-locked CSD
  - gamma shift w/ each cycle
MAXWELL EQUATIONS

Electrostatics, Magnetostatics

- **Faraday's law**
  - \( \nabla \times \mathbf{H} = \mu_0 \mathbf{J} \)

- **Divergence of curl**
  - \( \nabla \cdot \mathbf{B} = 0 \)

- **Divergence of divergence**
  - \( \nabla \cdot \nabla \mathbf{E} = \nabla \phi \)

Impressed currents
- currents due to ionic flow
- that "appear out of nowhere"
- (nearst batteries)

- **Faraday's law**
  - \( \nabla \times \mathbf{E} = \sigma \nabla \mathbf{H} \)

- **Gauss's theorem**
  - \( \nabla \cdot \mathbf{E} = \rho \)

- **Biot-Savart law**
  - \( \mathbf{B} \propto \nabla \times \mathbf{J} \)

N.B. these are all defined at a (every) point in space

Propagation of potentials, magnetic fields instantaneous (no capacitance)
- simultaneous eqs to solve  - \( \nabla \phi, \mathbf{B} \) are sources
- \( \nabla \mathbf{J}, \mathbf{E} \) are data

Linear
- potential (\( \phi \)) and magnetic fields (\( \mathbf{B} \)) produced by a weighted sum of two current source distributions are equal to weighted sum of fields produced by each current source distribution by itself.
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \]

(from Nunez, 1981)

\[ \vec{B} = \nabla \times \vec{A} \]

"vector potential"

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

\[ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \]

if linear in conductivity and dielectric, too, and fields periodic w/\( f \)

\[ \nabla \times \nabla \times \vec{E} = -2\pi f \mu \left( \sigma + 2\pi f \varepsilon \right) \vec{E} \]

to neglect:

\[ \frac{2\pi f \mu \left( \sigma + 2\pi f \varepsilon \right) \left| \vec{E} \right|}{\left| \nabla \times \nabla \times \vec{E} \right|} \ll 1 \]

1) \( \left| \nabla \times \nabla \times \vec{E} \right| \propto \left| \vec{E} \right| / L^2 \), where \( L \) is dist over which \( \vec{E} \) varies significantly

2) \( \mu \) & tissue similar to empty space

3) Assume conductor (large) \( \sigma \), dielectric unit, and EEG freq.

\( \xi \) number is about \( 10^{-6} \) \( \rightarrow \) small
**MONPOLE, DIPOLE FORWARD SOLN**

\[ \Phi_1 = \frac{s}{4\pi \sigma r} \]

Potential recorded for source monopole.

\[ \Phi_2 = \frac{s}{4\pi \sigma} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]

Potential recorded for source-sink pair ("near field").

Scalar \( \Phi_2 \approx \left( \frac{1}{4\pi \sigma} \right) \frac{s d \cdot \hat{r}}{r^3} \), \( r \gg d \)

Vector \( \vec{B}_2 \approx \left( \frac{\mu_0}{4\pi} \right) \frac{s d \times \hat{r}}{r^3} \), \( r \gg d \)

Approximations for "far enough away" measurements (subtracting two \( \frac{1}{r} \)'s gives inverse square).

\( \Phi_i(t) = e_i \cdot s_i(t) \)

Electrode gain over source strength.

\( \vec{b}_i(t) = \vec{m}_i \cdot s_i(t) \rightarrow \text{quadr. moment of } \vec{B} \)

\[ \Phi_i(t) = \sum_j e_{ij} s_j(t) \]

All sources.

\[ \vec{b}_i(t) = \sum_j m_{ij} s_j(t) \]

\( \geq 3 \) for \( \vec{B} \)

Linear superposition with fixed electrodes and sensors.

\[ \vec{X}(t) = \sum_j g_{ij} s_j(t) \]

Electromagnetic measures gain vector over strength.

\[ \vec{X}(t) = G \cdot \vec{s}(t) \]
Forward Solution

- well-posed (one answer)
- linear: \[ b(A) + b(B) = b(A + B) \]
- approximations due to unknown electrical properties of head

- 3-shell spherical analytic
  - skull & brain conductivity
  - "smearing" (cf. cable theory)

- 3-shell boundary element
  - solution = infinite homogeneous + finite element

- finite element
  - most general
  - computational intensive w/ small grid
  - many unknown parameters to estimate
\[
V_i = \sum_j E_{ij} S_j + \eta_i
\]

Matrix form:

\[
\begin{bmatrix}
V \\
\end{bmatrix} = \begin{bmatrix}
E \\
\end{bmatrix} S + \begin{bmatrix}
\eta \\
\end{bmatrix}
\]

lower case bold \( \rightarrow \) vector
upper case bold \( \rightarrow \) matrix

Electric recordings:

\[
\begin{bmatrix}
V \\
\end{bmatrix} = \begin{bmatrix}
E \\
\end{bmatrix} S + \begin{bmatrix}
\eta \\
\end{bmatrix}
\]

current
Source dipole
amplitudes
(same length
as above)

Magnetic recordings:

\[
\hat{x} = \hat{A} \hat{s} + \hat{n}
\]

Note: only one current source for each column in the \( E + B \) matrix!
WHY LOCALIZE?

- most of ERP literature based (indeed) on temporal "components"

but: 1) underlying local cortical generators (from microelectrode LFP, CSD)
   - extended in time (400 msec), visible from every scalp electrode
   - multiphasic in every cortical area
   - temporally non-static depending on stimulus
     "e.g. simple contrast, brightness diffs can modulate retinal delay by 50 msec!"

2) thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels

3) stimulus manipulations will change temporal overlap
   - may cause "component" peak to disappear without changing cortical areas being activated

4) verified by intra cortical LFP/CSD (Schroeder et al, 1998)

- by contrast, the spatial signature of the signal from any cortical area is static — a better area-based "component"

- temporal "components" should be retired (EEG started w/few electrodes, many time points)
- their original reason for being no longer relevant — easier to record now temporal points
  - easier to "paste" high level psychological functions onto a few waveform deflections

macaque monkey

Intracortical data

→ these areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey cortex

LFPs from approx. layer 4 in cortex ("input layer")

Psychologists now identify a few temporal "component" peaks...

→ but each peak comes from every one of these cortical areas!!
Derivation of Ill-posed Inverse

(from Dale & Sereno, 1993)

\[ x = As + n \]

\[ A = \text{forward soln matrix} \quad (E + B) \]

\[ s = \text{source vector} \]

\[ n = \text{sensor noise vector} \]

\[ Err_w = \left\langle \| Wx - s \|^2 \right\rangle \]

\[ \text{expectation: } \sum_k P_k K \]

\[ \text{assumed } n, s \text{ normal, zero-mean } w \text{ corresponding covar. matrices } C, R \]

\[ Err_w = \left\langle \| W(As+n) - s \|^2 \right\rangle \]

\[ = \left\langle \| (WA-I)s + Wn \|^2 \right\rangle \]

\[ = \left\langle \| Ms \|^2 + \| Wn \|^2 \right\rangle \quad \text{where } M = WA-I \]

\[ = \| Ms \|^2 + \| Wn \|^2 \]

\[ \quad \text{diag is noise variance (already squared)} \]

\[ = \text{tr}(MRMT) + \text{tr}(WCW^T) \]

\[ \quad \text{trace is sum of diag elements} \]

\[ \text{[re-expand]} \]

\[ = \text{tr} \left( WARA^TW^T - RATA^T - WAR + R \right) + \text{tr} (WCW^T) \]

Explicitly minimize by taking derivative w.r.t. \( W \), set to zero, solve for \( W \)

\[ 0 = 2WARA^T - 2RATA^T + 2WC \]

\[ WARA^T + WC = RATA^T \]

\[ W(ARA^T + C) = RATA^T \]

\[ W = RATA^T(ARA^T + C)^{-1} \]

\[ W \text{ is inverse solution operator: } \begin{bmatrix} \text{sensors} \\ \text{sources} \end{bmatrix} \rightarrow \begin{bmatrix} W \end{bmatrix} \]

Equivalence to minimum norm and Tikhonov regularized inverse if \( C, R \) are proportional to identity matrix (i.e., sensor noise & sources independent and equal variance).
inverse 2

\[ \text{INVERSE SIN} \] (2)

\[
W = RA^T(ARA^T + C)^{-1}
\]

\( \Rightarrow \) "minimum norm" solution

(find \( \hat{\beta} \) w/ smallest norm = \( \| \beta \| \))

- the minimum norm solution appropriately downplays deeper (= weaker scalp signal) sources since these are more likely to fall into the noise floor.

- "problems" of minimum norm:

  deeper sources get displaced to the surface

- small superficial sources "win" because of approx inverse square form of fwd solution
  \( \Rightarrow \) smaller norm of distributed superficial sol

- can't fix by increasing priors of deep sources!!
  \( \Rightarrow \) that will give deep sources given noise as input!!
INVERSE SOLUTIONS TO ILL-POSED COMPARED

\[ s = Wx \]

- How to use the inverse solution, \( W \)
- Same \( W \) for all time points

"minimum norm" solution

\[ \| W \| \] of solution is smallest of infinitely many alternate solutions

Linear inverse operator

\[ W = R A^T (A R A^T + C)^{-1} \]

\[ [W] = [R][A]^T \begin{bmatrix} A & R \end{bmatrix} [A^T] + [C] \]

\[ [A R A^T] \Rightarrow \text{Square in # of sensors (small)} \]

Alternate, algebraically equivalent Bayesian derivation (w/ bigger inverses!)

\[ W = (A^T C^{-1} A + R^{-1})^{-1} A^T C \]

\[ \begin{bmatrix} W \end{bmatrix} = \begin{bmatrix} A^T & \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} A \end{bmatrix} + \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \]

\[ \begin{bmatrix} A^T C^{-1} A \end{bmatrix} \Rightarrow \text{both square in # of sources (large)} \]
PROBLEMS W/ SURFACE NORMAL

- Since nearby points on surface often have different orientation, surface normal constraint can help (since fwd soln A, B very different)

- But, since point spread function typically extends across sulci, 'artifactual' sign reversals occur

- Solutions

1) Ignore sign \( \Rightarrow \) saves useful orientation info!

2) Solve onto 3 orthogonal dipoles at each critical point, instead of a single oriented dipole

\( \Rightarrow \) more appropriate when averaging across subjects, since detailed deviations vary a lot

\( \Rightarrow \) also, fills in bottom of sulcus (else unsigned stripes)
use inverse-2

FMRI Constrained Inverse

- insert FMRI values for Rii's
- but still allow other sites to have non-zero Rii's
- pathologies occur if solution restricted completely to FMRI points by setting non-FMRI Rii's to zero

- this allows extracting time course from sources visible in EEG/MEG and FMRI
- N.B.: sources that are only visible in EEG/MEG will be dispersed to small distributed values at a large number of vertices
  visible in both EEG/MEG and FMRI
  visible only in EEG/MEG and not FMRI
distributed at small amplitude across many vertices

[diagram of brain with arrows indicating source localization]
NOISE SENSITIVITY NORMALIZATION

forward: \( x = As \) well-posed (Liu, Dale, and Belliveau, 2002)
inverse: \( s = Wx \) ill-posed
solve: \( x = As + n \) for \( s \)

\[ W = R A^T (R A R^T + C)^{-1} \]

- multiply inverse operator by noise sensitivity matrix, \( D \) (diagonal)
  \[ D_{ii} = \frac{1}{\text{diag}(\sqrt{W}\ C\ W^T)} \]

\[ W_{\text{norm}} = DW \]

\[ s_{\text{norm}} = (W_{\text{norm}} x)_i = (DWx)_i = \frac{W_i x}{\sqrt{\text{W}_{\text{norm}} (W_{\text{norm}})^T}} \]

if assume Gaussian white noise, noise covariance, \( C \), is multiple of \( I \), so

\[ W_{\text{wgn}} = \frac{W_{\text{orig}}}{\| W_{\text{orig}} \|} \]

i.e., scale each row of \( W \) by single value — the norm of that row — row of \( W \) is:

\[ S_i = \frac{W_{\text{orig}} \cdot x}{\| W_{\text{orig}} \|} \]

inverse solution coefficient for \( s \) => scale (divide) by norm of this row

that is, if inverse sol’n for deep source is reduced by interaction of inverse square nature of feed and min norm, dividing by norm of row of inverse (same) will increase/rescue deep source.
Noise Sensitivity Normalization (2)

Shallow source (unit strength)
- \( W_\text{fwd big} \)
- \( \text{inv small} \)

Deep source (unit strength)
- \( W_\text{fwd small} \)
- \( \text{inv reduced because of minimum norm} \)
  \( \text{(N.B., W should be bigger than for superficial source)} \)
  \( \text{but, min norm reduces lot of inverse square)} \)

\[
S_i = \frac{\text{\( W_i \) orig} \cdot \hat{X}}{|| W_i \text{ orig} ||}
\]

- Shallow: small
- Deep: even smaller!

- Effect on inverse solution: more like significance is actual power
- Effect on point-spread function: to equalize shallow & deep

[Shallow spread out more than min norm]
[Deep shrunk to same as shallow]

Point-spread functions
- Noise not normalized
- Noise normalized
**Conclusions**

- More EEG or more MEG better.
- EEG better than MEG (cf. radial) (EEG forward currently less accurate)
- Biggest gain from adding small # EEG (n MEG) (e.g. 20)
  to many MEG (n EEG) (e.g. 150)
- Easier to add many MEG, so: optimal < 30 EEG
  < 300 MEG
- EEG/MEG forward-solution-scaling-factor error
  causes → more crosstalk
**music** (1)

(from Dale & Sereno, 1993) (cf. Mosher & Leahy)

- Using **sensor covariance**

\[
D = \langle xx^T \rangle = \sigma^2 I + \sum_i \sum_j \sigma_i \sigma_j \mathrm{Cov}(i,j) A_i A_j^T
\]

\[\sim \left[ x_1 \ldots x_n \right] \left[ x_1 \ldots x_n \right]^T \]

\[
\frac{1}{n} \text{ recording time points}
\]

\[
D = U \Lambda U^T = \left[ \begin{array}{c|c|c} & & \\
0 & 0 & \Lambda_1 \\
& \ddots & \ddots \\
& & \Lambda_n \\
\end{array} \right]
\]

**columns of U matrix are orthonormal basis vectors of "spatial pattern" space (one per is spatial pattern across sensors)**

- Find, order most significant **spatial patterns** in sensors over time

**Project** forward solns onto these spatial patterns

("project" = dot prod = similarity) for each point in brain

\[
\mathbf{z}_i = \mathbf{A}_i \mathbf{U}_i \Lambda \mathbf{U}_i^T \mathbf{A}_i^T \Rightarrow \text{big single number if forward soln looks like Us}
\]
MUSIC

\[ (2) \quad \text{how to weight the minimum norm inverse} \]

\[ R_{ii} \sim \frac{A_i^T A_i}{A_i^T U \Lambda U^T A_i} \]

\[ R = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \]

\[ W = R A^T (A R A^T + C)^{-1} \]

- like parallel resistance
\[ R_{\text{parallel}} = \frac{1}{1/R_1 + 1/R_2 + \ldots} \]

So any low resistance \( R_i \) decreases overall resistance (small \( R_{ii} \))

So i.e., if toward soln has appearance like any low eigenvalue spatial pattern, it gets devalued
- How it works: take advantage of spatial information that changes over time

- How it fixes min norm problem

N.B.: Problem if assumption about lack of perfect correlation is violated.

E.g., if two widely separated sources (i.e., diff. find solns) are highly correlated, MUSIC will eliminate both since no single fixed soln will look like that "2-separated dipole" pattern (e.g., L/R A=1).

"Dual MUSIC" back to fix...