RAFAEL E. NÚÑEZ, LAURIE D. EDWARDS and JOÃO FILIPE MATOS

EMBODIED COGNITION AS GROUNDING FOR SITUATEDNESS AND CONTEXT IN MATHEMATICS EDUCATION

ABSTRACT. In this paper we analyze, from the perspective of 'Embodied Cognition', why learning and cognition are situated and context-dependent. We argue that the nature of situated learning and cognition cannot be fully understood by focusing only on social, cultural and contextual factors. These factors are themselves further situated and made comprehensible by the shared biology and fundamental bodily experiences of human beings. Thus cognition itself is embodied, and the bodily-grounded nature of cognition provides a foundation for social situatedness, entails a reconceptualization of cognition and mathematics itself, and has important consequences for mathematics education. After framing some theoretical notions of embodied cognition in the perspective of modern cognitive science, we analyze a case study – continuity of functions. We use conceptual metaphor theory to show how embodied cognition, while providing grounding for situatedness, also gives fruitful results in analyzing the cognitive difficulties underlying the understanding of continuity.

1. Introduction

An important goal of mathematics education is to understand the thinking involved in doing and learning mathematics. In recent years, it has become widely accepted that the learning and practice of mathematics are not purely intellectual activities, isolated from social, cultural, and contextual factors (Lave, 1988; Collins et al., 1989; Cobb, 1994; Confrey, 1995). Instead, it has been acknowledged that learning and teaching take place, and have always taken place, within embedding social contexts that do not just influence, but essentially determine the kinds of knowledge and practices that are constructed (Lave and Wenger, 1991; Rogoff, 1990; Walkerdine, 1982). Perspectives that focus on the social and contextual nature of knowledge, and that make the embedding situation prominent in the analysis of cognition, have been labeled as 'situated'. Research and theoretical frameworks based on a situated approach to cognition insist that linguistic, social, and interactional factors be included in any account of subject matter learning, including the learning of mathematics. The hallmark of this approach is that it 'considers processes of interaction as basic and explains individual cognitions and other behaviors in terms of

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their contributions to interactive systems' (Greeno, 1997, p. 15). Lave and Wenger make this claim explicit when they state that 'there is no activity that is not situated,' and when they note the perspective's 'emphasis on comprehensive understanding involving the whole person rather than 'receiving' a body of factual knowledge about the world; on activity in and with the world; and on the view that agent, activity, and the world mutually constitute each other' (Lave and Wenger, 1991, p. 33).

These approaches have yielded many important results, and have helped to move the analysis of learning beyond a narrow focus on individual and 'internal' cognitive processes. Yet we would argue that the nature of situated learning and cognition cannot be fully understood by attending only to contextual or social factors considered as inter-individual processes. Thinking and learning are also situated within biological and experiential contexts, contexts which have shaped, in a non-arbitrary way, our characteristic ways of making sense of the world. These characteristic ways of understanding, talking about, and acting in the world are shared by humans by virtue of being interacting members of the same species, coexisting within a given physical medium. The overall aim of this paper is to present elements of a theory which focuses on how human cognition is bodily-grounded, that is, *embodied* within a shared biological and physical context, and to examine the ways in which this embodiment helps to determine the nature of mathematical understanding and thinking. In particular, we intend to investigate foundational aspects of situatedness by bringing in alternative approaches to orthodox cognitive science, approaches which focus on embodiment. One of our claims is that the situated cognition perspective, as valuable as it, leaves open important questions, such as: What is the grounding for situated knowing and learning? What is the basis of social situatedness? We share with the situated learning approach the belief that knowledge and cognition exist and arise within specific social settings, but we go on to ask what it is that makes possible the mutual intelligibility underlying shared social understandings. Our claim is that the grounding for situatedness comes from the nature of shared human bodily experience and action, realized through basic embodied cognitive processes and conceptual systems.

When taken seriously, genuine embodiment entails a reconceptualization of the nature of cognition and of mathematics itself, with corresponding implications for teaching (Lakoff and Núñez, 1997; forthcoming). A first implication is that we must leave behind the myth of mind-free mathematics as being about eternal, timeless truths, a legacy of Plato and Descartes. From an embodied perspective, the notion of an objective mathematics, independent of human understanding, no longer makes sense.

Another implication is that we are required to give an account of a mind-based mathematics, including an explanation of its stability and efficacy, in terms of the human bodily-based and situated conceptual systems from which it arises. Such an account should be useful in understanding problems in the teaching and learning of mathematics, and in designing more effective instruction.

In this paper, we will first frame the notion of embodied cognition within an intellectual and theoretical context, and elaborate the relationship between situated learning and embodied cognition. Next, we will present an example of a mathematical topic – continuity – that has presented teaching and learning difficulties, as a case study that can be fruitfully understood from an embodied cognition perspective. We close with a discussion of implications and directions for further work in mathematics education utilizing the perspective of embodied cognition.

2. Framing Embodiment Theoretically and Historically

Early mainstream cognitive science (cognitivism)

The situated learning perspective was welcomed by educational researchers and theorists as a richer and more appropriate means of addressing cognition than that offered by the formal, cognitivist models of early mainstream cognitive science, which emerged in the 1970s. This latter approach, strongly influenced by the objectivistic tradition of analytical philosophy and by functionalism, focused on learning as a process of individual reasoning, often explained in computational terms. Researchers in mainstream cognitive science at that time held that in explaining human cognition, it was necessary (and sufficient) to 'posit a level of analysis wholly separate from the biological or neurological, on the one hand, and the sociological or cultural, on the other' (Gardner, 1985, p. 6). This separate level of analysis focused on the individual as a processor of information, and characterized reasoning as the manipulation of arbitrary symbols. Under this view, symbols gain meaning from being associated with an objective reality, which is modeled in the mind by internal representations corresponding, to greater or lesser degrees of accuracy, with that external reality. This view of cognition became (and still is) pervasive in cognitive psychology (see for example, Sanford, 1985; Eysenk and Keane, 1992), and was subsequently adopted by some researchers in mathematics education seeking a paradigm for understanding mathematical thought.

This school of cognitive science – cognitivism – maintained the Cartesian dualism which holds that the mind is an abstract entity, separate from and transcending the body. Reasoning (including mathematical thought) is also non-corporeal, timeless, and universal. Concepts, the products of reasoning, are similarly abstract, and are not limited by physical or bodily realities. Cognitivism is thus based on objectivism, the doctrine that assumes transcendental ontological truths that are independent of human understanding (for details see Núñez, 1995). Under this view of knowledge, an objectivist would hold that the Pythagorean theorem, for example, *is* true and valid in this and in any universe, irrespective of the existence of human beings. This framing of mind and reasoning is consistent with a model of cognition as computation, and with a functionalist stance which holds that is it possible to study the mind purely in terms of the functions it performs, without seriously considering how the brain and body actually work.

The limitations, both theoretical and empirical, of cognitivism have become apparent in the 25 years since it became prominent. For example, this approach has been unable to satisfactorily model or account for everyday cognitive phenomena such as common sense, sense of humor, and natural language understanding. In addition, the information processing models that came out of early mainstream cognitive science bore little resemblance to the observed processes of real life problem-solving and learning found either inside or outside the classroom (Rogoff and Lave, 1984; Lave, 1988; Confrey, 1990; Nunes and Bryant, 1996). Furthermore, the objectivism of mainstream cognitive science was incompatible with the premises of radical constructivism, which does not assume a pre-determined reality that is straightforwardly accessed by the observer or learner (Cobb, 1994; Cobb, Yackel and Wood, 1992; Von Glasersfeld, 1990). As a result of these limitations, many researchers in mathematics education concerned with developmental, social, and cultural factors have rejected cognitive science, assuming that it had little to offer. However, alternative approaches to the scientific study of human mind have emerged within cognitive science itself, approaches that reject the assumptions of the objectivist, dulist, and functionalist school. In these approaches, cognitive processes and concepts are not abstract or transcendent, but rather fully embodied, emergent phenomena.

Embodied cognition

In the 1980's, a number of alternative perspectives emerged within cognitive science, originating in different disciplines, but sharing a commitment to investigating cognition as a physically-embodied phenomenon, realized via a process of codetermination between the organism and the medium in which it exists. Rather than positing a passive observer taking in a pre-determined reality, these paradigms hold that reality is constructed by the observer, based on non-arbitrary culturally determined forms of sense-making which are ultimately grounded in bodily experience.

The term 'embodiment' is used in a number of different ways within contemporary cognitive science, and these varied uses at times reflect fundamental theoretical differences. For some, embodiment refers to the phenomenological aspects of the human bodily experience (Merleau-Ponty, 1945; DiSessa, 1983), and the resulting psychological manifestations (Rosch, 1994). Certain theorists stress the unconscious aspects of bodily experience that underlie cognitive activity and linguistic expression (Johnson, 1987; Lakoff, 1987). Others focus on the organization of bodily action under principles of non-linear dynamics (Thelen and Smith, 1994). Yet others emphasize the biologico-structural codefinition that exist between organisms and the medium in which they exist, from which cognition results as an enactive process (Maturana and Varela, 1987). Along these lines, some stress the importance of the supra-individual biological processes that underlie high level cognition (Núñez, 1997), and others bring in embodiment as a crucial paradigm in anthropology (Csordas, 1994; Lock, 1993). Notions of embodiment are even explicitly used in the design of responsive and adaptive non-living systems (Brooks and Stein, 1993) and in structured connectionist computer models of cognitive linguistic activity (Feldman et al., 1996; Regier, 1996).

At a foundational level, our analysis builds on work by Rosch in cognitive psychology (Rosch, 1973, 1994; Varela et al., 1991); Edelman (1992) in neuroscience; Maturana and Varela in theoretical biology (Maturana and Varela, 1987) and more explicitly on the work by Lakoff and Johnson in cognitive linguistics (Lakoff and Johnson, 1980, 1998), and Lakoff and Núñez in mathematical cognition (Lakoff and Núñez, 1997; forthcoming). All these scholars share a focus on the intimate relation between cognition, mind, and living bodily experience in the world, that is, on the ways in which complex adaptive behavior emerges from physical experience in biologically-constrained systems.

Within this paradigm, the knower and the known are codetermined, as are the learner and what is learned. Thus, cognition is about enacting or bringing forth adaptive and effective behavior, not about acquiring information or representing objects in an external world. The potential of this perspective for building a more satisfactory account of human thinking is expressed by Varela, Thompson and Rosch, when they state, 'If we wish to recover common sense, then we must invert the representationist attitude [of a pre-given world] by treating context-dependent know-how not as a residual artifact that can be progressively eliminated by the discovery of more sophisticated rules, but as, in fact, the very essence of *creative* cognition' (Varela et al., 1991, p. 148).

3. Embodiment and the analysis of conceptual structures

Since the concept of embodiment is relatively new within the field of mathematics education, we would like to clarify our use of the term, and distinguish it from other notions concerning the role of the physical and concrete in mathematics learning. From our perspective, embodiment is not simply about an individual's conscious experience of some bodily aspects of being or acting in the world (e.g., memories of the first time we went skating or riding in a roller coaster). Embodiment does not necessarily involve conscious awareness of its influence. Nor does embodiment refer to the physical manipulation of tangible objects (e.g., playing with Cuisinaire rods or pattern blocks), or to the virtual manipulation of graphical images and objects instantiated through technology. Although there is a relationship between such experiences and the technical concept of embodiment, an embodied perspective does not constitute a prescription for teaching in a 'concrete' way. Similarly, although embodiment may provide a theoretical grounding for understanding the teaching and learning of 'realistic' or 'contextualized' mathematics, it is not directly concerned with 'contextualization' or 'situatedness' in subject matter teaching. Rather, embodiment provides a deep understanding of what human ideas are, and how they are organized in vast (mostly unconscious) conceptual systems grounded in physical, lived reality.

Johnson (1987) offers a nice example of how basic, universal bodily experience serves as the grounding for abstract understandings in his discussion of the experience of balance. The experience of balance is part of our everyday life and makes possible our physical experience of the world as well as our survival in it. The experience of being physically-balanced is so basic and pervasive that we are rarely aware of it. Balancing is an activity we learn with our bodies from very early ages, simply by acting, existing and developing in the world. It is not learned by acquiring abstract rules or algorithms. Moreover, the sensation of balance is so basic that all *homo sapiens* – no matter when and where they live on earth – have experienced it. As such, it is one of a class of deep, unconscious, yet pervasive bodily-based experiences providing a space of commonalties that makes up the ground for shared human sense-making. At the same time, the meaning of this experience, its working out in cultural expressions such as language, art, dance, science, and so forth, is both socially-constructed and situated.

We build up the meaning of balance through the active ongoing experience of bodily equilibrium and loss of equilibrium. Along with this process, we start making sense of related systemic bodily experiences; for example, the feeling that our fingers are not warm *enough*, or the mouth is *too* dry, and so on. Our understandings of 'too much', 'not enough', or 'out of balance' are pre-conceptual, non-formal, and non-propositional. Sense-making is built up in advance of formal or abstract concepts of quantity or 'balance'. The embodied meaning of balance is intimately related to our experience of bodily systemic processes and states of being in the world, and in particular, to the *image-schematic* structures that make those experiences coherent and significant for us.

Image-schemata

Image schemata are perceptual-conceptual primitives that allow the organization of experiences involving spatial relations. Some examples of image schemata are the *container* schema (which underlies concepts like IN and OUT); the *source-path-goal* schema (TO and FROM); the *contact* schema; and the *verticality* schema. Many basic concepts are built on combinations of these schemata. For instance, the English concept *on* uses three basic schemata: verticality, contact, and support. Image schemata appear to be universal, although in different languages the meanings of the words characterizing spatial relations may be composed of different combinations of these primitives. For example, not all languages have a single concept like the English *on*. In German, the *on* in *on the table* is rendered as *auf*, while the *on* in *on the wall*, which does not have the support schema, is translated as *an*. Thus the two German *on*s decompose into different combinations of the three component image schemata (Lakoff and Núñez, forthcoming).

It is important to mention that image-schemata are not static propositions that characterize abstract relations between symbols and objective reality. Rather they are dynamic recurrent patterns which order our actions, perceptions, and conceptions. These patterns emerge as meaningful structures for us mainly through the bodily experience of movement in space, manipulation of objects, and perceptual interactions. As Johnson states, 'They are a primary means by which we construct or constitute order and not mere passive receptacles into which experience is poured' (Johnson, 1987, p. 29).

Abstract concepts such as 'balancing' colors in a picture, 'balancing' a checking account, or 'balancing' a system of simultaneous equations are conceptual extensions of the image schemata involved in the bodily experience of 'balance'. These extensions occur through concepual mappings, including the important mechanism known as conceptual metaphor.

Indeed, it is our thesis that the basis of a great deal of our mathematical knowledge lies in such conceptual mappings. As Lakoff and Núñez state, 'much of what is 'abstract' in mathematics ... concerns coordination of meanings and sense making based on common image-schemata and forms of metaphorical thought. Abstract reasoning and cognition are thus genuine embodied processes' (Lakoff and Núñez, 1997, p. 30).

Conceptual metaphor

Conceptual metaphors are 'mappings' that preserve the inferential structure of a source domain as it is projected onto a target domain. Thus the target domain is understood, often unconsciously, in terms of the relations that hold in the source domain. For instance, within mathematics, Boolean logic is an extension of the container schema, realized through a conceptual metaphorical projection of the logic of containers. This metaphorical projection preserves the original inferential structure of IN, OUT, and transitivity, developed originally via physical experiences with actual containers, and later unconsciously mapped to a set of abstract mathematical concepts (Lakoff and Núñez, forthcoming).

The 'projections' or 'mappings' involved in conceptual metaphors are not arbitrary, and can be studied empirically and stated precisely. They are not arbitrary, because they are motivated by our everyday experience – especially bodily experience, which is biologically constrained. Research in contemporary conceptual metaphor theory has shown that there is an extensive conventional system of conceptual metaphors in every human conceptual system. These theoretical claims are based on empirical evidence from a variety of sources, including psycholinguistic experiments, generalizations over inference patterns, extensions to novel cases, historical semantic change, and the study of spontaneous gestures (Lakoff, 1993).

It has been found that metaphorical mappings are not isolated, but occur in highly-organized systems and combine in complex ways. As with the rest of our conceptual system, our system of conventional conceptual metaphors is effortless and lies below the level of conscious awareness (when we consciously produce novel metaphors, we utilize the mechanisms of our unconscious conventional metaphor system). Unlike traditional studies of metaphor, contemporary embodied views don't see conceptual metaphors as residing in words, but in thought. Metaphorical linguistic expressions thus are only surface manifestations of metaphorical thought (for an extensive discussion of conceptual metaphor theory and mathematics, see Lakoff and Núñez, forthcoming).

Embodiment and situatedness

The fact that embodiment and the mechanisms involved in conceptual mappings specify non-arbitrary links between cognition and experience helps to answer our question about the grounding for situated knowing and learning. That is, embodiment offers a rationale for the mutual understanding that exists within social situatedness. As Johnson says,

> Meaning is always a matter of human understanding, which constitutes our experience of a common world that we can make sense of. A theory of meaning is a theory of understanding. And understanding involves image-schemata and their metaphorical projections ... These embodied and imaginative structures of meaning have been showed to be shared, public, and 'objective', in an appropriate sense of objectivity (Johnson, 1987, p. 174).

From this point of view, cognition is neither subjective and isolated – unique to an individual – nor completely determined by external influences. Conventionalized meaning, although it is never context-free, depends to a great extent on shared image-schemata and conceptual projections, practices, capacities, and knowledge. Meaning is in many ways socially constructed, but, it is *not arbitrary*. It is subject to constraints which arise from biological embodied processes that take place in the ongoing interaction between mutually constituted sense-makers and the medium in which they exist (Núñez, 1997). Therefore, it is not surprising that cognition and learning are situated. Cognition is embodied; it is biologically grounded in individuals who interact with each other; hence it is also social and cultural.

4. Case study: Continuity of functions

It is widely accepted that teaching and learning the concept of 'continuity' of a function, so important for calculus, is a difficult task (Tall and Vinner, 1981; Robert, 1982; Núñez, 1993; Kitcher, 1997). The question then is, why is this the case? Is continuity *per se* a difficult concept? In this section we would like to illustrate how embodied cognition offers fruitful answers to these questions, by utilizing the tools of cognitive linguistics to analyze the concept (for details see Lakoff and Núñez, 1997, and Núñez and Lakoff, 1998).

Let us start taking a look at what textbooks say about continuity of a function. Here is a citation from a typical textbook introducing the concept:

> In everyday speech, a 'continuous' process is one that proceeds without gaps or interruptions or sudden changes. Roughly speaking, a function $y = f(x)$ is continuous if it displays similar behavior, that is, if a small change in x produces a small change in the corresponding value $f(x)$ ... Up to this stage, our remarks about continuity have been

rather loose and intuitive, and intended more to explain than to define (Simmons, 1985, p. 58).

Later in the same text, one finds what is called the 'rigorous', 'formal' and definitive definition of 'continuity' of a function, namely:

A function f is continuous at a number a if the following three conditions are satisfied:
1. f is defined on an open interval containing a,
2. $\lim_{x\to a} f(x)$ exists, and
3. $\lim_{x\to a} f(x) = f(a)$.
Where $\lim_{x\to a} f(x)$ (the limit of the function at a) is defined as:
Let a function f be defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement

$$\lim_{x\to a} f(x) = L$$

means that for every $\epsilon > 0$, there exists a $\delta > 0$, such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

This definition of continuity of a function – also called the Cauchy-Weierstrass definition – is said to be, and taught as, the definition that captures the essence of what continuity is. It is considered, and taught as, superior and more precise than the so-called 'intuitive' and 'informal' one. Moreover, as is evident in the text cited above, this definition intends more 'to define' than 'to explain'.

So far, this is the standard (and from our point of view, misleading, non-situated, disembodied) story. Let us step back, and carefully analyze what is going on, cognitively, when considering the statements and ideas involved in the two definitions.

The two definitions of continuity

The informal/intuitive definition that characterizes a 'continuous process as one that proceeds without gaps or interruptions or sudden changes' was used by such eminent mathematicians as Newton and Leibniz in the 17th century. Euler characterized a continuous function as 'a curve described by freely leading the hand'. This definition involves cognitive contents such as motion, flows, processes, change in time, and wholeness. These cognitive contents are the result of conceptual extensions from bodily grounded image-schemata and conceptual mappings that are natural to the human conceptual system. They are built on, among others, source-path-goal schemata, fictive motion metaphors, and basic conceptual blends (for details see Lakoff and Núñez, 1997). For these reasons, the textbook previously mentioned is correct in referring to this idea as occurring in 'everyday' speech. What Newton, Leibniz, and Euler did was simply (and, probably, unconsciously) apply the inferential structure of the everyday understanding of motion, flow, and wholeness, to a specific domain of human understanding: functions and variations. For the purposes of this case study, we will call this concept *natural continuity*.

The Cauchy-Weierstrass definition on the other hand, involves radically different cognitive content. It implicitly denies motion, flow and wholeness, dealing exclusively with static, discrete, and atomistic elements, which are conceptual extensions of rather different cognitive primitives, such as part-whole schemata and container schemata. The point is that, cognitively speaking, these two definitions are radically different; yet, *per se*, neither is superior to the other. Although it is true that the so-called 'rigorous' definition deals better with complex and 'pathological' cases (such as $f(x) = x \sin 1/x$) for certain purposes, it is not because it captures better the 'essence of continuity'. Within an embodied, non-objectivist cognitive science, there is no transcendental 'essence' of a concept, even in mathematics (Edwards and Núñez, 1995). It does so simply because it is built on a different collection of bodily grounded conceptual mappings that happen to deal well with both the prototypical cases of functions encountered prior to the 19th century (e.g., $f(x) = \sin x$; or $f(x) = 1/x$), as well as with the so-called pathological cases. This is the source of its utility and efficacy.

For the purposes of this article, the pedagogical problem can be summarized as follows: students are introduced to *natural continuity* using concepts, ideas, and examples which draw on inferential patterns sustained by the natural human conceptual system. Then, they are introduced to another concept – *Cauchy-Weierstrass continuity* – that rests upon radically different cognitive contents (although not necessarily more complex). These contents draw on different inferential structures and different entailments that conflict with those from the previous idea. The problem is that students are never told that the new definition is actually a completely different human-embodied idea. Worse, they are told that the new definition captures the essence of the old idea, which, by virtue of being 'intuitive' and vague, is to be avoided. This essence is usually understood as situation-free, that is, independent of human understanding, social activity, and philosophical enterprises.

Embodied cognition analysis of the two concepts

Let us analyze, from the perspective of embodied cognition, why these two concepts of continuity – natural and Cauchy-Weierstrass – are cognitively so different. Although it is not within the scope of this article to provide a full cognitive analysis of these two ideas (for a complete analysis, see Lakoff and Núñez, forthcoming), we will present several relevant aspects for the purpose of a deeper comparison. In particular we will focus on the fact that the term 'continuity', as used in mathematics, can refer to three distinct ideas. One is *natural continuity* (as in the 'informal', 'intuitive' definition) and the other two (implicit in the 'rigorous' Cauchy-Weierstrass definition) are *Gaplessness* (for lines as sets of points) and *Preservation of Closeness* (for functions).

Natural continuity:

The following are some essential features of a continuous function according to natural continuity:

a) the continuous function is formed by motion, which takes place over time.
b) there is a directionality in the function.
c) the continuity arises from the motion.
d) since there is motion, there is some entity moving (in Euler's version, the hand).
e) the motion results in a static line with no 'jumps'.
f) the static line that results has no directionality.

What is the source of these notions of motion, staticness, and directionality? From the perspective of embodied cognition, we conceive of the mobile and static aspects of a continuous curve via the activation of an everyday human conceptual process: the fictive motion metaphor (Talmy, 1988). This metaphor can be summarized as follows:

- A Line IS The Motion of a Traveler tracing that line.

Examples of this mapping are abundant in everyday language:

– Highway 80 *goes to* Sacramento.
– Just before Highway 24 *reaches* Walnut Creek, it *goes through* the Caldecott Tunnel.
– After *crossing* the bay, Highway 80 *reaches* San Francisco.

In these cases a highway, which is a static linear object, is conceptualized in terms of a traveler moving along the route of the highway. Using the same cognitive mechanism, we can speak, in mathematics, of a function as *moving, growing, oscillating, approaching values,* and *reaching* limits. It is worth noting that this way of speaking is not limited to students but includes professional mathematicians as well. Formally speaking, the function does not move, but cognitively speaking, under this metaphor, it does – and that is what matters in terms of understanding.

These embodied natural and everyday human cognitive mechanisms are the ones that make possible the intuitive dynamic and static conceptualization of a continuous function. As in Euler's characterization, continuity is

characterized by motion in the Fictive Motion metaphor. Using this methodology we can give a precise cognitive account of Euler's intuitive notion of continuity for a function in terms of elements of ordinary embodied human cognition, showing how mathematical ideas are constituted out of ordinary bodily grounded ideas.

Cauchy-Weierstrass continuity:

The Cauchy-Weierstrass definition was motivated by complex mathematical objects that mathematicians first encountered in the 19th century, and emerged from three important intellectual movements of that time: the arithmetization of analysis; the set-theoretical foundations movement; and the philosophy of formalism. These movements were separate in their goals, but complementary in their effects on the development of mathematics. All of them required conceptualizing lines, planes, and n-dimensional spaces as sets of points.

The Cauchy-Weierstrass definition requires a series of cognitive primitives, also embodied in nature, but different from the ones used to conceptualize natural continuity. There are at least three relevant conceptual metaphors that combine their inferential structures in a systemic way to give an extremely powerful mathematical tool. These metaphors are:

- A Line IS a Set of Points
- Natural Continuity IS Gaplessness
- Approaching a Limit IS Preservation of Closeness Near a Point

A Line IS a Set Of Points

In general terms, there are two importantly different ways of conceptualizing a line:
1) A holistic one, not made up of discrete elements, where a line is absolutely continuous and points are locations *on* a line. In this sense, a line is an entity distinct from the points, that is, locations on that line, just as a highway is a distinct entity from the locations on that highway. Lines, from the perspective of our everyday geometric intuition, are natural continua in this sense.
2) A Line Is A Set Of Points. According to this metaphor, the points are not locations *on* the line, but rather they are entities *constituting* the line.

The first characterization is congruent with natural continuity and the second with Cauchy-Weierstrass' definition. The distinction between these two ways of conceptualizing lines (and hence planes and n-dimensional spaces) has been crucial throughout the history of mathematics, and the failure to distinguish between them has led to considerable confusion. Both conceptions are natural, in that both arise from our everyday conceptual system. Neither is 'right' or 'wrong' per se; however, they have very different cognitive properties, and provide different inferential structure. It is this fact that should be taken into account in the teaching and learning process.

Natural Continuity IS Gaplessness

According to our everyday intuition, a line constitutes a *natural continuum*. As we move along a line, we go through point-locations. When we move continuously along a line from a location *A* to a location *B*, we go through all point-locations on the line between *A* and *B*, without skipping over any, that is, without leaving any gaps between the point-locations. In this case we will say that the collection of point-locations between *A* and *B* is *gapless* when the line segment *AB* is naturally continuous.

The metaphor underlying the Cauchy-Weierstrass definition identifies the point-locations on a line as constituting the line itself. Such a metaphorical 'line' *is not* a natural continuum, but only a set of points. When a naturally continuous line segment is conceptualized as a set of points, that set of points will be *gapless*. Thus, in this specific situated conceptual context, the metaphor A Line Is A Set Of Points entails the metaphor:

- Natural Continuity IS Gaplessness.

Therefore, a line conceptualized as a set of points cannot be – cognitively – *naturally continuous* but only *gapless*. This terminology thus distinguishes two distinct ideas, based on different cognitive mechanisms, that have previously both been called 'continuity'.

Approaching a Limit IS Preservation of Closeness Near a Point

In Cauchy-Weierstrass' definition of limit there is no motion, no time, and no 'approach'. Instead, there are static elements. The definition calls for a *gapless* 'open interval' of real numbers; there are no lines and no points and no surfaces in that metaphorical ontology for the Cartesian plane. The plane itself is a made up of a set of pairs of real numbers. The gaplessness of the set of real numbers in the open interval is Cauchy-Weierstrass' metaphorical version replacing the natural continuity of the intuitive line in Newton's geometric idea of a limit.

The idea of the function *f* approaching a limit *L* as *x approaches a* is replaced by a different idea (in order to arithmetize 'approaching' avoiding motion), that is, *preservation of closeness near a real number: f(x)* is arbitrarily close to *L* when *x* is sufficiently close to *a*. The epsilon-delta condition expresses this precisely in formal logic. What the Cauchy-Weierstrass approach does is to provide a new metaphor:

- Approaching A Limit IS Preservation Of Closeness Near A Point.

When Cauchy-Weierstrass 'define' continuity for a function, they cannot mean – cognitively – the natural continuity assumed by Newton for ordinary lines, that is, natural continua. Again, they must use conceptual mappings (metaphors) that allow them to reconceptualize geometry (holistic lines) using arithmetic (discrete numbers). Just as they needed a new metaphor for approaching a limit, they needed a new metaphor for continuity of a function. They characterize this new metaphor in two steps: first at a single arbitrary real number, and then throughout a (gapless) interval. Their new metaphor for continuity uses the same basic idea as their metaphor for a limit: *preservation of closeness.* Continuity at a real number is conceptualized as preservation of closeness, not just near a real number but also *at* it. Continuity of a function throughout an interval is thus preservation of closeness near and at every real number in the interval.

What is precise in the Cauchy-Weierstrass definition?

Textbooks and curricula lead students to believe that it is the epsilon-delta portion of these definitions that constitutes the rigor of this arithmetization of analysis. Moreover, they are led to believe that it is this aspect that captures the essence of what 'continuity' *is*. As we see, not only this is not true, but the epsilon-delta aspect of the definition actually plays a far more limited role. The epsilon-delta aspect accomplishes only a precise characterization of the notion 'correspondingly'. This notion occurs in the dynamic definition of a limit, where the values of *f(x)* get 'correspondingly' closer to *L* as *x* gets closer to *a*. This is the only vagueness that is made precise by the epsilon-delta definition.

Another interesting element in the Cauchy-Weierstrass definition is the role played by the idea of 'gaplessness'. The Cauchy-Weierstrass formulates the 'definition of continuity' with the explicit condition that the function is defined over an open interval. It assumes this open interval to be gapless. Since gaplessness was the way to metaphorically conceptualize continuity on the real line, it assumes a gapless (i.e., 'continuous') input to the function. What this definition really shows is that (1) when these metaphors hold, especially when lines are metaphorically conceptualized as sets of real numbers, and (2) when the input of the function is gapless, and (3) when the function preserves closeness, then (4) the output is also gapless.

Why has it been widely accepted that Cauchy-Weierstrass' definition of preservation of closeness was instead a 'definition of continuity'? The answer is that it has been assumed, falsely, that Cauchy-Weierstrass' metaphors capture the essence of continuity because they deal effectively, for the purposes of the arithmetization program, with prototypical and pathological cases. Given the metaphor that a line is a set of real numbers, then natural continuity can only be conceptualized metaphorically as gaplessness. Since Cauchy-Weierstrass' open interval condition guaranteed that the inputs to the function are always gapless, it is no surprise that preservation of closeness for a function with a gapless input guarantees a gapless output. If the input is metaphorically continuous (that is, gapless), then the output is going to be metaphorically continuous (gapless). Since the metaphors are mostly realized through unconscious processes, and they fit the prototypical cases, they are not noticed as being metaphorical or controversial in any way. Furthermore, since the open interval condition hid the continuity (gaplessness) required in the input, Cauchy-Weierstrass' definition appeared even to the originators to be a definition of continuity, when in fact, all it did was guarantee that a gapless input for a function gives a gapless output.

5. Discussion

As a discipline, mathematics education is concerned not only with creating effective means and methods of instruction, but with understanding why certain methods are effective and others are not, and with larger questions about the nature and development of mathematical knowledge. Our answers to these questions, and even the ways we choose to investigate them, are strongly influenced by our implicit or explicit conceptualization about the nature of human thought, and about mathematics itself. When mathematics is conceived of as an external realm of objective truths, to be 'discovered' through the application of rational thinking, then the investigation of mathematics learning focuses on accurate mappings, models, and internal representations of mathematical entities and relationships. If, on the other hand, mathematics is conceived as a product of adaptive human activity in the world, shared and made meaningful through language, but based ultimately on biological and bodily experiences unique to our species, then mathematics education must take a different approach. New practices in mathematics education, from classroom teaching to scientific research and curriculum design, should emerge that present mathematics as a genuine mind-based activity with all its embodied peculiarities and beauty.

Through the analysis of the idea of continuity, we have seen that certain ways of talking and thinking about mathematics can be misleading, with unfortunate pedagogical consequences. These consequences can arise when we ignore how our conceptual system works, implicitly assuming the existence of a 'mind-free' mathematics. We propose that one important source of pedagogical problems in mathematics education are the philosophical foundations that have dominated our view of mathematics (objectivism, platonism, formalism). These philosophical commitments are necessarily (if unintentionally) transmitted in the teaching process, which can lead to the teaching of definitions and supposed eternal truths that capture mathematical essences, rather than mind-based, embodied, human forms of sense-making. The fundamental conceptual error underlying this kind of teaching is the idea that intuition can be replaced by rigor in order to eliminate vagueness. Not only is this not possible, but it is not necessary for effective learning (c.f., Smith et al., 1993/94). If one studies natural, situated, spontaneous, everyday thoughts and intuitions from an embodied cognitive perspective, one finds that they are not at all vague. The tools provided by the embodied cognition approach allow one to characterize precisely how the inferential structure of everyday bodily experience, which underlies intuition, is mapped onto more abstract domains.

Basic mathematical ideas show an impressive stability over hundreds, sometimes thousands of years. For this to happen requires, on the one hand, a common set of neural and bodily structures with which to contruct mathematical concepts. On the other hand, it requires that this conceptual construction make use of the most commonplace of everyday experiences, such as motion, spatial relations, object manipulation, space, and time. The study of the conceptual structure of mathematics from an embodied point of view shows how mathematics is built up out of such informal, everyday experiences and ideas. For this reason, mathematics cannot be conceived as a pure and 'abstract' discipline. Our mathematical conceptual system, like the rest of our conceptual system, is grounded in our bodily functioning and experiences. Seen from this perspective, situated cognition is not about 'situating' mind-free truths in meaningful contexts, but rather about examining how the human creation of mathematics arises from sense-making which is not arbitrary precisely *because* it is bodily grounded.

This view has important entailments for mathematics education. Rather than looking for better ways to help students learn 'rigorous' definitions of pre-given mathematical ideas, we need to examine the kinds of understanding and sense-making we want students to develop. We should look at the everyday experiences that provide the initial grounding for the abstractions that constitute mathematics. This is not necessarily an easy undertaking, since the grounding structures are often unconscious and taken-for-granted. At times, this grounding can be found in immediate physical experience, as in the case of work with early arithmetic, space, size, and motion. At other times, the grounding for a mathematical idea takes place indirectly, through a chain of conceptual mappings whose nature may be obscured by conventional language, but which can be revealed by utilizing the analytic tools of contemporary embodied cognitive science. In either case, what is important is to re-examine mathematical ideas in order to create instruction that complements the ways our conceptual systems naturally work.

In addition, we should provide a learning environment in which mathematical ideas are taught and discussed with all their human embodied and social features. Students (and teachers) should know that mathematical theorems, proofs, and objects are about ideas, and that these ideas are situated and meaningful because they are grounded in our bodily experience as social animals. Providing an understanding of the historical processes through which embodied ideas have emerged can support this aim. This does not mean simply presenting a few names and dates as a prelude to teaching the 'real' mathematics. It means talking about the motivations, zeitgeist, controversies, difficulties, and disputes that motivated and made possible particular developments in mathematics. Pierpont (1899), for example, provides excellent material for appreciating the controversies surrounding intuition, the concept of continuity, and the arithmetization program at the turn of the century.

In this paper, we have attempted to provide an overview of the essential elements of a theory of embodied cognition, and to apply this relatively new framework to the analysis of mathematical thought and learning. This framework is extremely rich, and systematic work on the analysis of mathematical thought is only beginning to take place. We presented a brief account of one such analysis, in order to illustrate the potential of this framework. We also addressed the relationship between theories which emphasize the socially-situated nature of cognition and the embodied cognition approaches. From our perspective, there is no contradiction between these approaches; rather, an understanding of the fundamental embodiment of cognition helps us to see how human beings are able to construct mutual understandings through social interaction. Since we, human beings, are all living physical creatures, acting within the same medium and sharing a basic biological heritage, we naturally experience the world in fundamentally similar ways. The conceptual structures which emerge in the human mind to make sense of our bodily experiences provide the raw material

for the construction of shared communication through language, and, subsequently, the shared construction of meanings. Thus, our understandings of the world, and of mathematics, may be socially and culturally situated, but is the commonalties in our physical embodiment and experience that provide the bedrock for this situatedness.

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RAFAEL E. NÚÑEZ
Institute of Cognitive Studies,
Berkeley, CA 94720,
U.S.A.
E-mail: Nunez@cogsci.berkeley.edu

LAURIE D. EDWARDS
School of Education,
St. Mary's College of California,
Moraga, CA 94575,
U.S.A.

JOÃO FILIPE MATOS
Department of Education,
University of Lisbon,
Portugal