CHAPTER

3

How Much Mathematics Is "Hardwired," If Any at All

Biological Evolution, Development, and the Essential Role of Culture

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INTRODUCTION

The investigation of mathematical cognition in developmental psychology has been primarily confined to the study of numerical cognition—basic facts involving quantity, such as the acquisition of number words and some competences for numerical estimation and calculation. But mathematics is much more than simple counting numbers and basic arithmetic. Mathematics, seen by many as the queen of sciences, is a unique and peculiar body of knowledge. It is objective, precise, relational, stable, robust, consistent, and highly effective in modeling aspects of the world—a feature expressed a century ago by the Hungarian physicist Eugene Wigner as the "unreasonable effectiveness of mathematics" (Wigner, 1960). Importantly, mathematics is quintessentially abstract—the very entities that constitute this domain are idealized

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mental abstractions that cannot be observed directly through the senses. A Euclidean point, for instance, is dimensionless and cannot be actually perceived. Even, say, the number "eight" cannot be observed directly, as we might perceive a collection we label as having eight glasses or eight chairs, but we do not perceive the number "eight" as such. Moreover, mathematics is a conceptual system that relies almost entirely on written practices—notations and symbols that make it happen. Cultures with no writing traditions or record-keeping practices may have some basic counting systems, but beyond that they have virtually no mathematics. Finally, mathematics is special because, after all, it is a strange nonintuitive conceptual system that often contains less than obvious concocted facts that are taught dogmatically and that usually remain unexplained. For instance, why does the multiplication of two negative numbers yield a positive result? Or why is the empty set a subset of every set? Or if $2! = 1 \times 2$, $3! = 1 \times 2 \times 3$, and so on, why then is $0! = 1$? None of these mathematical facts seems to find an instantiation in the natural world, and yet mathematics is full of these purely imaginary (and seemingly arbitrary) "truths" that, through formal definitions, axioms, and algorithms, provide stability and consistency throughout its conceptual edifice.

So, what kind of thing is mathematics then? What makes it possible? Can the study of the nature of mathematics inform research in developmental psychology? Is there any core of mathematics that is, as some neuroscientists and developmental psychologists proclaim, "hardwired" in the human brain? In this chapter, I address these questions and analyze the claim that humans are biologically endowed with some kind of protomathematical apparatus. To make the enterprise tractable within the confines of this chapter, I focus on the mathematical concept of the number line, as it provides an excellent case study for the investigation of these matters. The number line—a conceptual tool that allows for numbers to be conceived as locations along a line mapping numerical differences onto differences in spatial extension, as they appear in a ruler (technical details will be given below in the section entitled "Aren't Number-to-Space Mappings 'Hardwired'?")—is mathematically simple, yet it is extraordinarily powerful, and it is cognitively sophisticated enough to provide insight into the issues at stake. Prototypically, the number line has a linear mapping (i.e., with regular "steps" between successive whole numbers), but it could also have a nonlinear mapping as in a slide rule, which, having a logarithmic mapping, exhibits "steps" between numbers that get compressed as the number increase in magnitude. I evaluate the widespread assumption in contemporary research in the psychology and neuroscience of numerical cognition that the mental representation of the number line (linear or logarithmic)—the *mental number line*—is somewhat innate, "hardwired" in the human brain. More specifically, I analyze two central underlying tenets of this nativist position: (1) that the representation of number is inherently *spatial* (as a *line* is) and (2) that the number-to-space mapping—essential for the number line concept—is a universal intuition rooted directly in the brain through biological evolution, independent of culture, language, and education.

I begin by describing the major ideas involved in these nativist claims, pointing to some of the problems involved. I then review material from the history of mathematics, in particular from Old Babylonian mathematics, and from specific developments in 17th-century European mathematics, that is at odds with the nativist position. I follow the historical analysis with a review of results from recent empirical studies in experimental psychology as well as in cross-cultural cognition conducted with the Mundurukú from the Amazon and the Yupno from Papua New Guinea. These results show that the representation of number is not inherently spatial and that the intuition of mapping numbers to space is not universal, thus challenging the nativist claims. Taking the historical observations and experimental results together, I argue that even an idea as fundamental as number-to-space mapping—let alone the technical concept of number line or any more advanced mathematical concept—is far from having evolved via natural selection, requiring substantial cultural mediation and scaffolding for it to occur. I close the chapter with a discussion on the problems brought by the teleological structure of the nativist arguments, which occlude the constitutive role that culture plays in mathematical concepts. Using an analogy taken from culturally generated and sustained motor activity—snowboarding—I analyze how the investigation of the origin and nature of the concept of the number line shows the essential role of culture in high-level cognition and how this informs the understanding of the interplay between cultural processes and biological mechanisms in developmental theory.

NATIVISM IN COGNITIVE DEVELOPMENT, COGNITIVE NEUROSCIENCE, AND ANIMAL COGNITION

There is a widespread view, inspired by nativist ideas, that sees the origin of mathematical entities in biological evolution, existing independently of cultural and linguistic dynamics. This quote expresses the position clearly: “Mathematical objects may find their ultimate origin in basic intuitions of space, time, and number that have been internalized through millions of years of evolution in a structured environment and that emerge early in ontogeny, independent of education” (Dehaene, Izard, Spelke, & Pica, 2008a, p. 1217). From the same perspective, and with respect to the number line, specific claims have been made in cognitive neuroscience: “Different studies reporting introspective descriptions, behavioral data, and neuroimaging data support the assumption of a continuous, analogical, and left-to-right-oriented mental number line representing numbers, which is localized in the intraparietal sulcus” (Priftis, Zorzi, Meneghello, Marenzi, & Umiltà, 2006, p. 680). Not only is the mental line said to be directly “represented” in the brain without cultural, linguistic, or educational influences; these authors go further to make specific anatomical claims about where exactly and in what “direction” the line is represented.

Similar positions are defended in cognitive development and even in animal cognition. With respect to the more fundamental number-to-space mappings, a line of work has claimed that such mappings are manifested spontaneously in human infants and young children, as proclaimed by papers with titles reading “Spontaneous Mapping of Number and Space in Adults and Young Children” (de Hevia & Spelke, 2009) and “Number-Space Mapping in Human Infants” (de Hevia & Spelke, 2010). And, going phylogenetically much farther, recent titles in animal cognition reports have announced “Rhesus Monkeys Map Number onto Space” (Drucker & Brannon, 2014) and even “Number-Space Mapping in the Newborn Chick Resembles Humans’ Mental Number Line” (Rugani, Vallortigara, Priftis, & Regolin, 2015). What is the rationale of these nativist claims, and why do they look so appealing? And, importantly, do the data support these claims? (This latter question is addressed in the section entitled “Aren’t Number-to-Space Mappings ‘Hardwired’?” p. 90.)

For a start, there is abundant research suggesting that space might be a natural grounding for number representation. After all, space provides countless metaphors for number that pervade the history of modern mathematics (Lakoff & Núñez, 2000). And, more specifically, supported by distinct spatial brain areas (Feigenson, Dehaene, & Spelke, 2004; Göbel, Calabria, Farnè, & Rossetti, 2006; Zorzi, Priftis, & Umiltà, 2002), space plays a major role in number processing (Bächtold, Baumüller, & Brugger, 1998; Dehaene, Bossini, & Giraux, 1993; Gevers, Reynvoet, & Fias, 2003; for a meta-analysis, see Wood, Willmes, Nuerk, & Fischer, 2008). Crucially, linear space is said to be readily employed by children (Booth & Siegler, 2006; Siegler & Booth, 2004) when confronted with the number-line task, in which they are asked to report numerical estimations on a line segment. (We come back to this later in the subsection entitled "Are Number Mental Representations Inherently Spatial?", p. 100.)

The tradition of studying number–space relationships is in fact long. Consistent with descriptions already made by Galton in 1880, that numbers are pictured on a line, behavioral studies in the 1990s have shown that there are associations between spatial-numerical response codes (SNARC), which appear to have a specific left–right horizontal spatial orientation (in people who grow up in cultures with left-to-right writing traditions): Western participants are faster to respond to large numbers with the right hand and to small numbers with the left hand (Dehaene et al., 1993). Such results have led to the claim "that the core semantic representation of numerical quantity can be linked to an internal 'number line,' a quasi-spatial representation on which numbers are organized by their proximity" (Dehaene, Piazza, Pinel, & Cohen, 2003, p. 498). And neuropsychological studies with unilateral neglect patients (with right parietal lesions) appear to support this idea. When asked to bisect a line segment, these patients tend to locate the middle point farther to the right (ignoring the left space). When asked to find the middle of two orally presented numbers, they answer with numbers greater than the correct answer (Priftis et al., 2006; Zorzi et al., 2002). Similar responses were observed in healthy participants when functioning in corresponding brain areas was momentarily disrupted (Göbel et al., 2006). Moreover, for several decades, lesion studies have revealed the involvement of the parietal cortex in number processing

(Gerstmann, 1940) and the systematic activation of the parietal lobes during calculation (Roland & Friberg, 1985), facts that have been confirmed with positron emission tomography (PET) (Dehaene et al., 1996) and functional magnetic resonance imaging (fMRI) studies (Rueckert et al., 1996). Taken together, these findings have led to the proposal that the parietal lobes contribute to the representation of numerical quantity on a mental number line (Dehaene et al., 2003; Dehaene & Cohen, 1995). Consistent with these claims, the mental number line is routinely invoked when interpreting a variety of results in numerical cognition, from early approximate numerosity discrimination abilities (Feigenson et al., 2004) to simple "subtractions" in preverbal infants and nonhuman primates (Dehaene et al., 2003), to lateralized behavioral biases in newborn chicks driven by brain asymmetries (Rugani et al., 2015). As a result, generalized statements that "to compare numeric quantities, *humans* make use of a 'mental number line' with smaller quantities located to the left of larger ones" (Doricchi, Guariglia, Gasparini, & Tomaiuolo, 2005, p. 1663; emphasis added) or even that, in the animal (tetrapod) kingdom at large, "spatial mapping of numbers from left to right may be a universal cognitive strategy available soon after birth" (Rugani et al., 2015, p. 536) are taken for granted and go largely unchallenged.

In sum, the central idea in these nativist positions in cognitive neuroscience, cognitive development, and animal cognition is that there is a "core" in human cognition that is *mathematical* per se (the mental number line, in this case): a sort of mathematical embryo, which in itself is culture-, language-, and education-free. At this point it is therefore advisable to ask—leaving the number line and its underlying number-to-space mappings aside—whether with respect to *quantity* alone, there are any "hardwired" capacities at all.

QUANTITY-RELATED "HARDWIRED" CAPACITIES? YES, … ARE THEY MATHEMATICAL? NO

Quantity-related capacities have been empirically investigated for more than half a century. Experimental psychologists in the late 1940s began to document the capacity for making quick, error-free, and precise judgments of the quantity of items in arrays of up to three or four items. They called this capacity *subitizing*, from the Latin word for "sudden" (Kaufmann, Lord, Reese, & Volkmann, 1949). Since these experimental findings many studies have shown that humans (and other species) possess a sense of quantity observable at an early age, before a demonstrable influence of language and schooling. For instance, at 3 or 4 days, a baby is able to discriminate between collections of two and three items (Antell & Keating, 1983) and between sounds of two or three syllables (Bijeljac-Babic, Bertoncini, & Mehler, 1991). Under certain conditions, they can distinguish three items from four (Strauss & Curtis, 1981; van Loosbroek & Smitsman, 1990). By 4.5 months, babies exhibit behaviors that some have interpreted as having a rudimentary understanding of elementary arithmetic as in "one plus one is two" and "two minus one is one" (Wynn, 1992). And as early as 6 months, infants are able to discriminate between large collections of objects on the basis of approximate quantity, provided that they differ by a large ratio (8 versus 16 but not 8 versus 12; Xu & Spelke, 2000).

In sum, there is solid evidence showing that humans, and many nonhuman animals (Mandler & Shebo, 1982), do indeed possess some quantity-related innate capacities involving numerosity discrimination. Within a small numerosity range they are precise, as in subitizing, and approximate, as in the case of large numerosity comparison. But the crucial question is: Are these numerosity-related capacities *mathematical*? Do these phenomena constitute *mathematics* as such? If not mathematical proper, it is indeed tempting to call these capacities *proto*mathematical, or *early*-mathematical, or *pre*mathematical. But doing so has the problem of invoking profoundly misleading teleological arguments, which I analyze at the end of this chapter. For the moment, suffice to say that numerosity discrimination has been found even in mosquitofish, which has led scholars in animal cognition to interpret results as "spontaneous number representation" in fish (Dadda, Piffer, Agrillo, & Bisazza, 2009). Such moves bring the risk of ascribing "numerical" properties like precision, order, compositionality, or operativity to far simpler abilities that only involve stimuli discrimination, paving the way for a teleological argument that thousands of species, from fish to humans, have "*number* representations" as a result of biological evolution. Whether to call these capacities *mathematical* is not a mere harmless superficial semantic matter, as doing so brings important (misguided) theoretical consequences. As mentioned, mathematics is, among others, precise (not just approximative), relational, combinatorial, consistent, symbolic and abstract. Numerosity-related capacities in themselves do not posses these properties: they are not mathematical as such. Subitizing and large (approximate) numerosity discrimination seem to be innate, and they may provide necessary *early cognitive preconditions* (not precursors!) *for numerical abilities* (Núñez, 2009), but they are not *numerical* (or mathematical) as such, in the same way that an infant's first walking steps may provide necessary *early motor preconditions for snowboarding abilities*, but they are not *about* snowboarding as such (more on this at the end of the chapter).

AREN'T NUMBER-TO-SPACE MAPPINGS "HARDWIRED"? NO

As mentioned, recent reports in developmental psychology and animal cognition have claimed that number–space mappings are manifested spontaneously in human infants and young children without the influence of culture (de Hevia & Spelke, 2009, 2010). And, going farther, that such mappings manifest even in monkeys (Drucker & Brannon, 2014) and newborn domestic chicks (Rugani et al., 2015). Wouldn't this constitute evidence that mathematical properties, such as number–space mappings, are indeed hardwired, providing the building blocks for the number line? It would. Except that the evidence reported in those studies is not about spontaneous number (or numerosity)–to–space *mappings* (Núñez & Fias, 2015). The study with monkeys (Drucker & Brannon, 2014) investigates a particular instance of a space-to-space (not numerosity-to-space) mapping following training. And the studies with infants (de Hevia & Spelke, 2009, 2010) and newborn chicks (Rugani et al., 2015) do investigate numerosity but do not provide evidence of *mappings*. Rather, at best, they show that there might be *associations* between (or biases in) numerosity and space. Once again, as with the case of the number–numerosity distinction, we need conceptual and terminological clarity in order to avoid confusion when addressing the phenomena under investigation. The notion of mapping is essential for investigating the number-line concept, so we must have a clear definition of it.

When applied to numbers, a standard definition of mapping, as it appears in the *Encyclopædia Britannica* (2005), is: "any prescribed way of assigning to each object in one set a particular object in another (or the same) set. Mapping applies to any set: a collection of objects, such as all whole numbers, all the points on a line, or all those inside a circle." And the notion of mapping can be used to characterize the number-line concept with two defining criteria (Núñez, Cooperrider, & Wassmann, 2012):

(i) there is an actual mapping of each individual number (or numerosity) under consideration (e.g., counting numbers within a certain range) onto a specific location on the line, and that
(ii) this mapping defines a uni-dimensional space representing numbers (or numerosities) with a metric (at least approximative)—a distance function.

That the mapping defines a space with a metric means, in the case of a straight line, that it constitutes a one-dimensional space (i.e., a line) representing numbers with a translation-invariant Euclidean metric: the length of the segment spatially representing the difference of two numbers satisfies the properties of a Euclidean distance function in a one-dimensional space, which is invariant under addition (if the mapping is logarithmic, these properties apply on a log-transformed space). The number line, whether linear or logarithmic, thus has essential *mapping* properties that go well beyond the numerosity–space associations reported for preschool children and infants (de Hevia & Spelke, 2009, 2010) or for newborn chicks (Rugani et al., 2015). Such associations have been interpreted as evidence of a number-to-line "mapping," yet they do not assign to each number (or numerosity) in the collection under consideration a particular location in space (criterion i) and thus do not constitute actual number-to-space mappings, let alone define a space with a metric (criterion ii). Indeed, the reported number–space associations in preschool children (de Hevia & Spelke, 2009) were obtained on the basis of a bisection task in which participants were asked to indicate the midpoint of a line segment flanked by the numbers (or numerosities) "2" and "9." In that study a number–space association was said to exist if the mean reported midpoint exhibited a slight bias toward the number (or numerosity) "9." The reported associations for infants (de Hevia &

Spelke, 2010) were obtained by means of a looking habituation paradigm. A number–space association was said to exist if infants transferred the discrimination of ordered series of numerosities to the discrimination of an ordered series of line segments. The associations reported for newborn chicks (Rugani et al., 2015) were based on a left–right binary exploration choice. After being trained with a target numerosity (5 elements for some chicks, 20 for others), the chicks explored an environment containing two panels—to the left and to the right, displaying identical numerosities either smaller or greater than the target (2 or 8 elements, and 8 or 32, respectively). An association (interpreted as "mapping") was said to exist given that around 70% of the time the chicks preferred the left panel when the numerosity was smaller than the target and the right one when it was greater. (Importantly, the relative contrast with 8 elements—greater than target 5 but smaller than target 20—was not tested within individuals but between groups).

Despite the use of number–space "mapping" in the titles and argumentation of these reports, no actual number (or numerosity)–to–line mappings were investigated therein. The reported associations simply do not conform to standard definitions of mapping. The phenomena investigated in those reports are, at best, *biases* or *associations*, which indeed may manifest quite spontaneously in infants, and they may provide early cognitive preconditions (not precursors!) for a later consolidation of a mental number line (if necessary cultural conditions are provided). But "mappings" they are not. Mappings, which are essential for the constitution of the number-line concept, may very well build on these early associations—like a snowboarder builds on his/her early balance and locomotion abilities—but they seem to require more than pure biological endowment.

WHAT CAN WE LEARN FROM THE HISTORY OF MATHEMATICS? A LOT

If we want to understand the origin of the concept of number line, an important source of information is found in the history of mathematics. Here I review two relevant sources, Old Babylonia and the mathematics of the 17th century in Europe.

Numbers and Calculations without Number Lines in Old Babylonian Mathematics

If the mental number line is as fundamental and "hardwired" as claimed, we should expect ubiquitous manifestations of number lines throughout human history. Leaving aside the fact that far from all human groups have ever developed numerical and arithmetic systems (most never did!), let alone left behind observable evidence of the use of number lines, we can evaluate the innate mental number line claim being maximally permissive with regard to the hypothesis and consider exclusively civilizations known to have developed sophisticated arithmetical knowledge, such as Mesopotamia, ancient Egypt, and China. At least in these old civilizations the evidence of the use of number lines should be overwhelming, as it is the case in our modern society where we see number lines in rulers, calendars, and electronic devices. But no such corresponding evidence seems to exist, not even in these mathematically sophisticated civilizations. We know from ancient clay tablets in Mesopotamia, for example, that Old Babylonians developed highly elaborated knowledge of arithmetical bases, fractions, and operations without the slightest reference to number lines (Núñez, 2008). There are roughly half a million published cuneiform tablets, from which no more than 5,000 tablets contain mathematical knowledge. Only about 50 tablets have diagrams on them (Robson, 2008), but none provides evidence of number lines. As the historian of Mesopotamian mathematics Eleanor Robson (written personal communication, March 10, 2009) puts it, "There [were] no representations of number lines [in Babylonia]: that metaphor was not part of the Babylonian repertoire of mathematical cognitive techniques. Until very late indeed (3rd century BC) number was conceptualized essentially as an adjectival property of a collection or of a measured object." Some of the diagrams on the tablets, as in the famous YBC 7289 tablet, written around the first third of the second millennium BC (see Figure 3.1), do show lines with numbers associated with measurements. But contrary to what has been claimed (Izard, Dehaene, Pica, & Spelke, 2008), this does not imply that the makers and users of such tablets operated with a number-line concept. Indeed, historians of mathematics have cautiously pointed out the risks of providing "inventive" interpretations of old mathematical documents through the lenses of modern concepts (Fowler, 1999) and have wisely suggested

Figure 3.1 A clay tablet known as YBC 7289 from the Old Babylonian period, from the Yale Babylonian Collection. The tablet, from the first third of the second millennium BC, is one of the few containing drawings. It shows numbers associated with measurements but no depiction of number lines. According to contemporary historians of Mesopotamian mathematics, Old Babylonians did not operate with number-line concepts despite manifesting sophisticated notions of number and arithmetic.

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that the interpretation of such documents must be made in their historical context (Fowler & Robson, 1998; Robson, 2002).

The tablet YBC 7289 shows a square with its diagonals, the number 30 on one side, and the numbers 1, 24, 51, 10 and 42, 25, 35 written against one diagonal. For decades scholars have interpreted this pair of numbers to be, respectively, a four-sexagesimal-place approximation of $\sqrt{2}$ and the length of the diagonal, and have taken this as an "anachronistic anticipation of the supposed Greek obsession with irrationality and incommensurability" (Robson, 2008, p. 110). But while analyzing this tablet in context with other documents, known practices, and archaeological findings, Robson and colleagues have observed that the text is most likely a school exercise by a trainee scribe who got the approximations from a reference list of coefficients (Fowler & Robson, 1998). The round shape of the tablet, typically used by trainee scribes at that time, and the unusually large handwriting (characteristic of trainees) support the idea that the text was not written by a competent scribe but by a student. As part of a series of drills, the trainee scribe simply found the length of the diagonal by multiplying 30 (the side) by the pregiven constant 1, 24, 51, 10, exactly as indicated in a well-known coefficient list of the time, the tablet YBC 7243 (Robson, 2008). In fact, “there is no evidence that Old Babylonian scribes had any concept of irrationality” (p. 110). Similarly, another tablet—Plimpton 322, the most famous Babylonian mathematical artifact—was long thought to have been a trigonometric table. But recent detailed analyses of related tablets show that Old Babylonians did not operate with the concept of radius for calculating areas (they used $A = c^2/4\pi$, where c is the perimeter, instead of $A = \pi r^2$). Therefore, there was no conceptual framework for measured angle or trigonometry: Plimpton 322 cannot have been a trigonometric table (Robson, 2002). Tablets such as YBC 7289 and Plimpton 322 are nonnarrative and have pictorial or tabular components that can be inadequately read “as ‘pure,’ abstract mathematics, enabling them to be represented as artifacts familiar to [modern] mathematicians” (Robson, 2008, p. 288).

In sum, in the absence of a clear number-line depiction and narrative, simply because we see numerals, magnitudes, and lines on clay tablets, we cannot anachronistically infer that Old Babylonians operated with a number-line mapping or with a mental number-line representation. If, as experts say, Babylonians conceptualized number essentially as an adjectival property of a collection or of a measured object, we cannot conclude on the basis of YBC 7289 that Old Babylonians operated with a fundamental number-to-line mapping. In a nutshell, just because we observe people making adjectival statements like “plastic chair” and “wooden table,” we cannot conclude that they operate with a fundamental material-to-furniture mapping. Explicit characterizations of the number line seem to have emerged many centuries later in Europe, as late as the 17th century, and only in the minds of a few pioneering mathematicians.

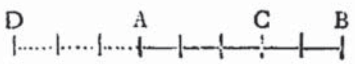
How Long Does It Take for the Number Line to Be Invented? A Long Time

According to the known historical record, it was apparently John Wallis in 1685 who, for the first time, introduced the concept of number line for operational purposes in his *Treatise of Algebra* (see Figure 3.2).

CHAP. LXVI. *Of Negative Squares.* 265

Yet is not that Suppoſition (of Negative Quantities,) either Unuſeful or Abſurd; when rightly underſtood. And though, as to the bare Algebraick Notation, it import a Quantity leſs than nothing: Yet, when it comes to a Phyſical Application, it denotes as Real a Quantity as if the Sign were +; but to be interpreted in a contrary ſenſe.

As for inſtance: Suppoſing a man to have advanced or moved forward, (from A to B,) 5 Yards; and then to retreat (from B to C) 2 Yards: If it be asked, how much he had Advanced (upon the whole march) when at C? or how many Yards he is now Forwarder than when he was at A? I find (by Subducting 2 from 5,) that he is Advanced 3 Yards. (Becauſe + 5 − 2 = + 3.)



But if, having Advanced 5 Yards to B, he thence Retreat 8 Yards to D; and it be then asked, How much he is Advanced when at D, or how much Forwarder than when he was at A: I ſay − 3 Yards. (Becauſe + 5 − 8 = − 3.) That is to ſay, he is advanced 3 Yards leſs than nothing.

Which in propriety of Speech, cannot be, (ſince there cannot be leſs than nothing.) And therefore as to the Line AB *Forward*, the caſe is Impoſſible.

But if (contrary to the Suppoſition,) the Line from A, be continued *Backward*, we ſhall find D, 3 Yards *Behind* A. (Which was preſumed to be *Before* it.)

And thus to ſay, he is *Advanced* − 3 Yards; is but what we ſhould ſay (in ordinary form of Speech, he is *Retreated* 3 Yards; or he wants 3 Yards of being ſo Forward as he was at A.

Which doth not only anſwer Negatively to the Queſtion asked. That he is not (as was ſuppoſed,) Advanced at all: But tells moreover, he is ſo far from being Advanced, (as was ſuppoſed,) that he is Retreated 3 Yards; or that he is at D, more Backward by 3 Yards, than he was at A.

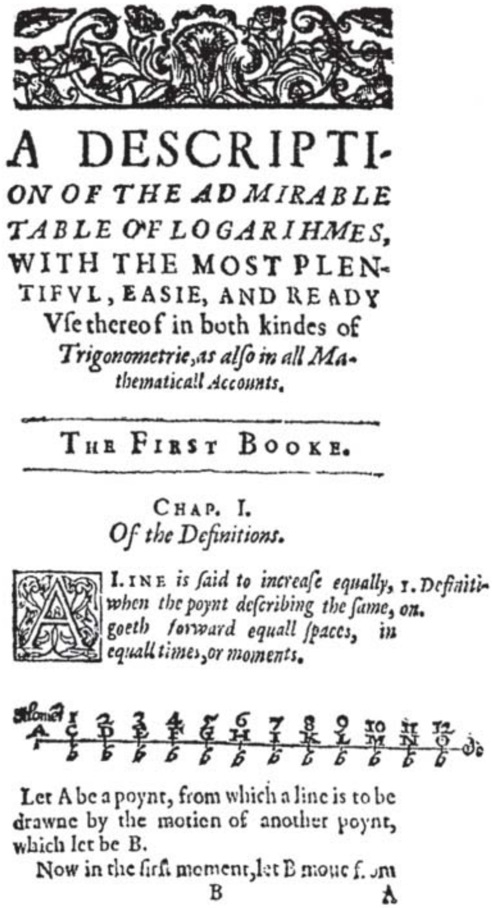
And conſequently − 3, doth as truly deſign the Point D; as + 3 deſigned the Point C. Not Forward, as was ſuppoſed; but Backward, from A.

So that + 3, ſignifies 3 Yards Forward; and − 3, ſignifies 3 Yards Backward: But ſtill in the ſame Streight Line. And each deſigns (at leaſt in the ſame Infinite Line,) one Single Point: And but one. And thus it is in all Lateral Equations; as having but one Single Root.

Now what is admitted in Lines, muſt on the ſame Reaſon, be allowed in Plains alſo.

Figure 3.2 The introduction of the number line in 17th-century Europe. The figure shows the top of page 265 of John Wallis's *Treatise of Algebra*, published in 1685, chapter 66, "Of Negative Squares and Their Imaginary Roots." This passage seems to be the first explicit characterization of a number line for operational purposes in the history of mathematics.

There he begins by introducing a metaphor in which a man moves forward and backward a certain number of yards and establishes basic arithmetical operations where numbers are conceived in terms of motion along a path. To an educated person alive today, the language and the explanations used seem strikingly childish and simplistic, akin to how the number line is taught today in elementary schools. In any case, Wallis may have built on earlier precursors who paved the way, such as John Napier, who, with his 1616 diagrams (originally published in Latin in 1614) used an explicit number-to-line mapping to define the concept of logarithm (see Figure 3.3).


A DESCRIPTI-
ON OF THE ADMIRABLE
TABLE OF LOGARIHMES,
WITH THE MOST PLEN-
TIFVL, EASIE, AND READY
Vse thereof in both kindes of
Trigonometrie, as also in all Ma-
thematicall Accounts.

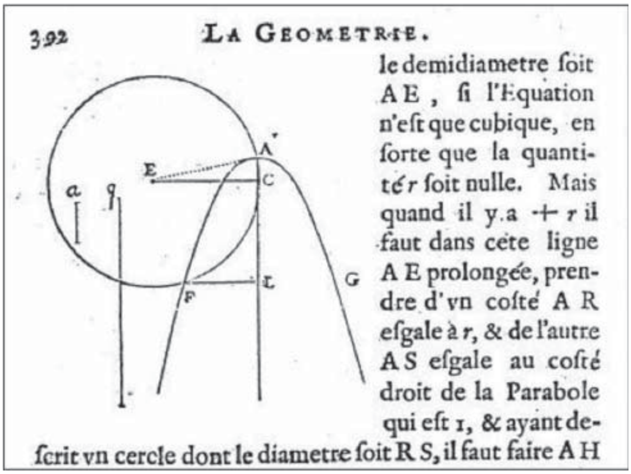
THE FIRST BOOKE.

CHAP. I.
Of the Definitions.

A LINE *is said to increase equally, when the poynt describing the same, goeth forward equall spaces, in equall times, or moments.* 1. *Definition.*

Let A be a poynt, from which a line is to be drawne by the motion of another poynt, which let be B.
Now in the first moment, let B moue from
B A

Figure 3.3 The first page of chapter 1 of John Napier’s *A Description of the Admirable Table of Logarihmes* published in 1616 (English edition), in which he introduces his definition of logarithm via an explicitly depicted number-to-line mapping.


392 LA GEOMETRIE.
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n'eſt que cubique, en
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té r ſoit nulle. Mais
quand il y a + r il
faut dans cete ligne
A E prolongée, pren-
dre d'vn coſté A R
eſgale à r, & de l'autre
A S eſgale au coſté
droit de la Parabole
qui eſt 1, & ayant de-
ſcrit vn cercle dont le diametre ſoit R S, il faut faire A H

Figure 3.4 A diagram from page 392 of Descartes's first edition of *La Géometrie*, published in 1637. Contrary to what has been claimed, Descartes did not introduce the number line in this classic volume through the invention of coordinate systems.

The number-to-line mapping, however, was not a common idea among mathematicians. Contrary to what has been claimed (Izard et al., 2008), Descartes did not introduce the number line through the invention of coordinate systems in his 1637 *La Géometrie* (1637/1954). Descartes's original 120-page text never mentions the number-line concept, and none of its 49 illustrations depicts a number line or a numerical coordinate system, not even when specific values for specific magnitudes are characterized. Figure 3.4 shows one of those illustrations, where the relevant magnitudes "*a*" and "*q*," which could have been displayed along a single encompassing left-to-right horizontal number line—as we would do today—are in fact depicted separately and vertically. Similarly, other such diagrams in Descartes's text, displaying up to four different magnitudes, depict the extensions separately, never along the same line. Descartes's classic text does not contain any specific numbers mapped onto lines but only numerically unspecified magnitudes in the tradition of Greek geometry.

It is important to point out that Wallis's and Napier's texts, intended for readers with advanced knowledge in mathematics, proceed with detailed and careful—almost redundant—explanations of how to construct and use a number-line mapping. These explanations are not "formalizations" of the idea of a number line, but rather, they are elaborated presentations

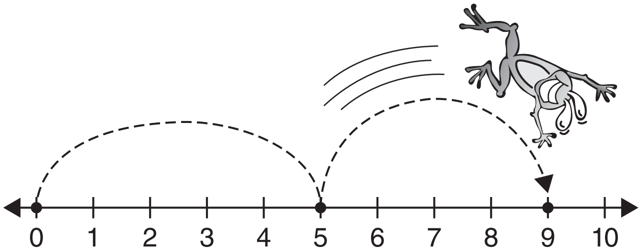

Figure 3.5 An illustration of the number line from a manual from the National Council of Teachers of Mathematics (in the United States) intended for children in kindergarten to fifth grade. The manual includes practice questions, such as "If I take a hop of 5 and then a hop of 4, where will I land?"

Source: http://illuminations.nctm.org/Lesson.aspx?id=355, Reprinted with permission.

of a new meaningful and fruitful idea that emerges from the demands for grounding abstract concepts that had turned too elusive—negative squares for Wallis and logarithms for Napier. The hand-holding narrative, however, is similar to what we see in many elementary school classrooms today (see Figure 3.5), showing just how unfamiliar the idea of a number line was to the elite 17th-century mathematicians, let alone to the rest of the majority of illiterate citizens in Europe at that time.

Taken together, these facts from the history of mathematics—from Old Babylonia to 17th-century Europe—are simply at odds with the idea of a "hardwired" mental number line that would spontaneously manifest itself in all humans. Of course, these facts by themselves do not prove that there was no mental number line before the 17th century, but they make the claim of an innate and spontaneous number-to-line mapping highly implausible. Moreover, supporters of such nativist claim must explain: (1) how Babylonian, Greek, and early Renaissance mathematicians in Europe built so much mathematics without a number-line concept; (2) why it took so long for professional mathematicians (let alone ordinary people) to come up with a notion of number line; and, crucially, (3) why the *la crème de la crème* of 17th-century European mathematicians had to go into such simplistic, childlike language to explain how to make use of it.

We now turn from these episodes in the history of mathematics, to contemporary experimental studies in number-to-space mappings, which have produced results that are consistent with the historical records.

ARE RESULTS IN EXPERIMENTAL STUDIES ON NUMBER-LINE MAPPINGS CONSISTENT WITH HISTORICAL RECORDS? YES

During the last decade or so, a significant number of studies have been carried out investigating children's mappings of numerical estimations onto lines. Developmental research has shown that when asked to locate numbers on a line segment marked with a 0 on the left and 100 on the right—*the number-line task*—even kindergarteners seem to readily place smaller numbers at left of the segment and greater numbers at right (Booth & Siegler, 2006; Siegler & Booth, 2004). Interestingly, they allocate more space to small numbers and less to big numbers in a nonlinear—logarithmic-like—compressed manner. The data support the view that numerical estimation follows the domain-general psychophysical Weber–Fechner law that subjective sensation increases proportional to the logarithm of the stimulus intensity. With schooling and mathematical training, the development of mapping patterns starts to shift gradually, between kindergarten and fourth grade, from a logarithmic-like pattern to a primarily linear one (Booth & Siegler, 2006; Siegler & Booth, 2004). Based on the assumption that number mappings are indicators of mental representations (Dehaene et al., 2008a; Priftis et al., 2006; Zorzi et al., 2002), variations of this number-line task have been used to investigate the properties of number and numerosity representations (e.g., linear mappings versus logarithmic mappings) in children of various ages and in populations from isolated cultures. In this section we review research that uses the number line task and addresses two main questions relevant to the theme of this chapter:

1. Are number mental representations (numerical magnitude) inherently spatial?
2. Is the intuition of mapping number to space hardwired and universal?

Are Number Mental Representations Inherently Spatial?

The number-line task has proven to be very useful for the investigation of numerical estimations and number representations in children (and adults). However, little attention has been paid to the fact that the results have been obtained with participants reporting magnitude on a *line*, which is an inherently *spatial* medium. It seems straightforward that a thorough investigation of the nature of number representation must disentangle the number–space confound intrinsic to number-line methods, dissociating number stimulus from reporting condition.

More than half a century ago, classic work in psychophysics documented nonlinear compressions of nonspatial stimulus sensory scales that included magnitude production (Stevens & Mack, 1959) and revealed a systematic relationship between them and numerical categories (Stevens, Mack, & Stevens, 1960). Number cognition research, however, obsessed with number-line associations, has largely neglected the study of nonspatial mappings (with rare exceptions, as in Vierck and Kiesel, 2010). Moreover, it has expected spatial representation to be de facto involved in number-related experimental tasks, even when the data suggest no involvement of spatial representations (see Fischer, 2006, for a discussion). Moving in a different direction, recent work in cognitive neuroscience has advanced the idea that number representation may build on a more fundamental magnitude mechanism (Cohen Kadosh & Walsh, 2008; Walsh, 2003), which can be studied nonspatially and may provide new insights into the understanding of the nature of number representation.

In a recent experimental study, Núñez, Doan, and Nikoulina (2011) investigated number mappings using spatial and nonspatial reporting modes in college-level-educated young adults. This population, which has already experienced the reported shift from logarithmic to linear mappings, exhibit differential linear-log mappings that vary according to stimulus modalities (Dehaene et al., 2008a). That is, when reporting on a line, educated adults exhibit logarithmic mappings when responding to number stimuli that are difficult to count or discern (e.g., tones and large numbers of dots) but produce linear mappings when responding to symbolic stimuli (e.g., words) or to small nonsymbolic number stimuli presented visually (e.g., small numbers of dots). In this study, Núñez and colleagues reasoned that if number representation builds not just on spatial resources but on deeper magnitude mechanisms, one should expect these differential mappings to be similarly enacted with nonspatial reporting conditions. The study, including symbolic and nonsymbolic number/numerosity stimuli modalities, considered three nonspatial reporting conditions: two manual–instrumental ones where participants squeezed a dynamometer and struck a bell with various intensities, and one noninstrumental condition, where they vocalized with various intensities.

The study replicated previous results with educated participants who had reported spatially (Dehaene et al., 2008a): nonlinear compression for hard-to-count nonsymbolic stimuli (Figures 3.6N and 3.6O), and linear mappings for easy-to-count 1–10 dots and symbolic stimuli (Figures 3.6M, 3.6P). But the authors found that when number estimations were reported nonspatially, the mappings elicited by nonsymbolic stimuli were

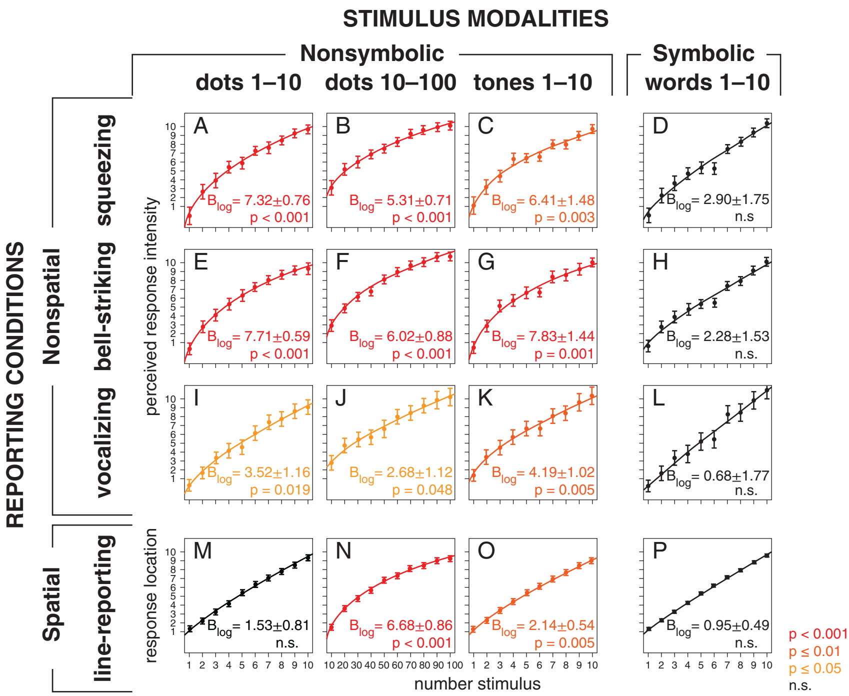

Figure 3.6 College-level students' responses to a number mapping task, by stimulus modality and reporting condition. Data are mean perceived response intensity (nonspatial reporting) and mean response location on the line (spatial reporting), with corresponding standard errors of the mean. B_{log} indicates the relevant unstandardized weight (plus/minus standard error) of the logarithmic regressor in a multiple regression analyses with linear and logarithmic predictors. Gray-scaled graphs indicate a significant *p*-value for the corresponding B_{log} weight (based on a B_{log}/(standard error) *t* ratio with df = 7 in each case; black indicates a nonsignificant *p*-value).

Source: Reprinted from Núñez et al., 2011, with permission.

consistently nonlinear throughout (Figures 3.6A–C; 3.6E–G; 3.6I–K). When reporting nonspatially, linearity only occurred as a response to symbolic stimuli (words; Figures 3.6D, 3.6H, 3.6L). Crucially, while logarithmic-like mappings were manifested consistently across all non-symbolic stimulus modalities, spatial reporting produced linear mappings in the key case of 1–10 dots (Figure 3.6M).

Current proposals that attempt to explain the logarithmic-to-linear shift in people with scholastic training (Dehaene et al., 2008a) cannot explain these results. One proposal says that logarithmic representations gradually vanish and become linear with education but that they remain "dormant" for very large numbers or whenever there is number approximation. A second proposal states that experience with measurement and with the invariance principles of addition and subtraction—essential features of number concepts—may play an important role in the shift. But the results of the study do not support these arguments. First, logarithmic thinking in educated participants, far from being "dormant" or vanishing, did manifest consistently in *all* nonspatial reporting conditions— instrumental and noninstrumental. It is worth pointing out that peripheral sensorial compression, which account for magnitude effects in sensory neurons (Nieder & Miller, 2003), cannot explain by itself the nonlinearity patterns observed with nonspatial reporting, since these reporting conditions systematically produced linear responses for symbolic stimuli (words). Second, when reporting nonspatially, college students—who are intensely exposed to measurement practices as well as to addition and subtraction—did not map nonsymbolically presented numerosities in a linear manner. When reporting nonspatially, they even failed to produce a linear mapping in the simple 1–10 dots case. The invariance principle, which profiles the equality of numerical magnitude between predecessors and successors of a given natural number (e.g., the absolute difference between 6 and 5 is the same as the absolute difference between 6 and 7), seems to apply only when participants report on a line, where linear mappings are produced. But reports via squeezing, bell-striking, or vocalizing produced logarithmically compressed mappings that do not hold invariance since allocated reporting magnitude units were larger for small number stimuli than for big ones. It is as if the difference between "3" and "2" and the difference between "9" and "8" are judged as equal when reporting on a line but not when reporting nonspatially where the former difference is reported to be larger.

In the Núñez et al. (2011) study, the only case of linearity manifested in 1–10 dots was observed with spatial reporting (Figure 3.6M). This case of linear mapping, although widely studied, appears to be an exception rather than the norm. This particular case may indeed be explained by exposure to measurements—a culturally shaped activity through which a fixed unit is applied to different spatial locations (thus implementing and embodying the invariance principle). Since logarithmic mappings are preferred by children (Booth & Siegler, 2006; Siegler & Booth, 2004), the nonlinear representations of nonsymbolic quantities seem to be—even in the college-educated mind—not just "dormant" but the most fundamental ones. The details of neural organization in the primate brain may provide (at least partial) support for logarithmic encoding. For instance, behavioral and neuronal representations of numerosity in the prefrontal cortex of rhesus monkeys exhibit a nonlinearly compressed scaling of numerical information as characterized by the Weber–Fechner law and Stevens's power law for psychophysical magnitudes (Nieder & Miller, 2003). This is consistent with the fact that in this study, numerical estimation by college-educated participants followed these laws only when the stimuli were primarily nonsymbolic (numerosities), but not when they were symbolic (number words). The monkey results show that the compressed scale is the natural way that numerosity is encoded in the primate brain. It has been argued that this is the case in the language-free monkey brain (Dehaene et al., 2003). But the results in the Núñez et al. study (2011) suggest that, when nonsymbolic stimuli (numerosities) are concerned, this might be the case even in the human college-educated brain *with* language. The differential linear/nonlinear patterns produced by symbolic and nonsymbolic stimuli make it unlikely that number/numerosity representation is determined uniquely by the internal organization of cortical representations (Dehaene et al., 2003), requiring the participation of other neural pathways and brain dynamics involved with culturally mediated symbolic and language processing (Ansari, 2008).

In the study by Núñez and colleagues (2011), linear mappings manifested only in specific culturally supported cases: symbols (words) and small quantities of dots when they were reported on a notational device—the line segment. The case of 1–10 dots is particularly revealing, for it picks out four essential features present in the culturally mediated learning of the powerful concept of the number line: (i) nonsymbolic material, presented in (ii) small quantities of (iii) visually perceivable (iv) discrete objects. The initial exposure to and learning of the number line in early school years is done, precisely, in the presence of these essential features: by mapping onto the horizontal line quantities representing discrete small amounts of visually perceivable items, such as single-digit counts of hops (Ernest, 1985; and see Figure 3.5). The remaining stimulus modalities lack these four features: 10–100 dots may be visually perceivable, but they are not readily countable or discernible; tones are not perceived visually; and words are symbolic. It is a fact of our cultural educational practices that the learning of the number line is not done with tones or with large numbers. Moreover, learners are never (or hardly) exposed to mapping numbers onto modalities such as squeezing, bell-striking, and vocalizing. This lack of exposure and experience is reflected in the levels of imprecision of nonspatial responses in the Núñez et al. study whose variability across participants was systematically higher than in the ubiquitously trained spatial one. Thus, culturally mediated symbolic-linguistic dimensions may account for the only linear mappings observed in this study. That is, they account for the specificity of linear reports which were exclusively given in response to culturally created symbols (number words) across all reporting conditions —spatial and nonspatial—and to 1–10 dots when reported on a culturally created notational device: the line segment.

Now, if the linear properties of number mappings do not seem to be the most fundamental ones, why is the number-to-line mapping so powerful and ubiquitous? The reason seems to be that the spatial medium offers unique affordances and advantages over other media. When using marking devices, number-to-space reporting can be stored, readily shared with others, and inspected days, months, or years later. Moreover, building on the specificities of the primate's body morphology and brain, which is largely devoted to visual processing and object manipulation, the spatial medium allows for close visuomanual monitoring, precision, and metacognitive processing. These advantageous features—absent, or weak, in nonspatial reporting—seem to have been culturally selected and privileged, eventually leading to the elaboration of number-line mappings mediated via notational and measuring devices (Ifrah, 1994) and to the complexities of number-line and graphic displays that are ubiquitously present in the modern world.

In sum, the answer to the main question in this subsection—"Are Number Mental Representations Inherently Spatial?"—seems to be "No." The results reviewed here suggest that number representation is not inherently spatial but rather that it builds on a deeper magnitude sense that manifests spatially and nonspatially mediated by number/numerosity magnitude, stimulus modality, and reporting condition. Number-to-space mappings—although ubiquitous in the modern world—do not seem to be part of the human biological endowment but have been culturally privileged and developed.

Is the Intuition of Mapping Number to Space "Hardwired" and Universal?

The majority of the research in human number cognition has been carried out with educated participants, mostly from cultures with writing systems that make extensive use of number-to-space mappings. Since little (or almost nothing) is known about the psychology and neuroscience of number mappings in societies with no writing and measurement practices, it is highly likely that the theoretical accounts we have today are based on a biased sample. Hence the importance of investigating such populations before their cultural and language traditions are absorbed by the ever-increasing world globalization. In this section we review two such studies, one conducted with the Mundurukú from the Amazon (Dehaene et al., 2008a) and one with the Yupno of the remote mountains of Papua New Guinea (Núñez et al., 2012).

Unintended Results with the Mundurukú of the Amazon: Failing to Exhibit Number-Line Mappings

The Amazonian Mundurukú is a group known for having a language with a reduced lexicon for precise numbers—1 through about 5 only (Pica, Lemer, Izard, & Dehaene, 2004; Strömer, 1932)—and little exposure to education and measuring devices. Since the Mundurukú can operate with sophisticated quantity and spatial concepts in an approximate and nonverbal manner (Dehaene, Izard, Pica, & Spelke, 2006; Pica et al., 2004), Dehaene and collaborators adapted Siegler and colleagues' number-line task (Booth & Siegler, 2006; Siegler & Booth, 2004; Siegler & Opfer, 2003) for their purposes using symbolic (words) and nonsymbolic stimuli (dots and tones) in the 1–10 range (Dehaene et al., 2008a). Stimulus numbers "1" and "10" were used and presented as anchors at the left end and the right end of the line segment, respectively. After running the task, the authors reported that "the Mundurukú's mean responses revealed that they understood the task. Although some participants tended to use only the end points of the scale, most used the full response continuum and adopted a consistent strategy of mapping consecutive numbers onto consecutive locations" (p. 1217). The authors further reported that Mundurukú mapped symbolic (words) and nonsymbolic (dots, tones) numerosities onto a logarithmic scale, while Western control participants used linear mapping with small or symbolic numbers and logarithmic mapping when numerosities were presented nonsymbolically. They concluded that (a) the concept of a linear number line is a product of culture and formal education and (b) the mapping of numbers onto space (with metric) is a universal intuition and this initial intuition of number is logarithmic. Beyond the reported logarithmic compressions of participants' judgments, which may be explained in light of the psychophysical Weber–Fechner law, the data seem to support (a), which is consistent with the findings in the study (Núñez et al., 2011) analyzed in the previous subsection. But, ironically, a careful analysis of the Mundurukú study shows that the data in fact do not support the most important claim (b)—that the mapping of numbers onto space is a universal intuition. What is more, these data support the very opposite claim: that the mapping of numbers onto space (with metric) is *not* a universal intuition. How is this possible?

The Supporting Online Material of the Mundurukú study mentions (but, surprisingly, does not analyze) that *some* participants tended to produce "bimodal" responses using only the endpoints of the line segment—not the full extent of the response continuum (Dehaene et al., 2008b). According to Dehaene et al.'s own words, the number-line task has a fundamental criterion that "participants evaluate the size of the numbers and place them at spatial distances relative to the endpoints that are proportional to their psychological distances from those endpoints" (Dehaene, Izard, Pica, & Spelke, 2009, p. 38c). Bimodal responses, however, are categorical responses that violate this fundamental criterion since they do not reflect psychological distances with a metric—the "distance function" of criterion (ii) mentioned in p. 91—and therefore

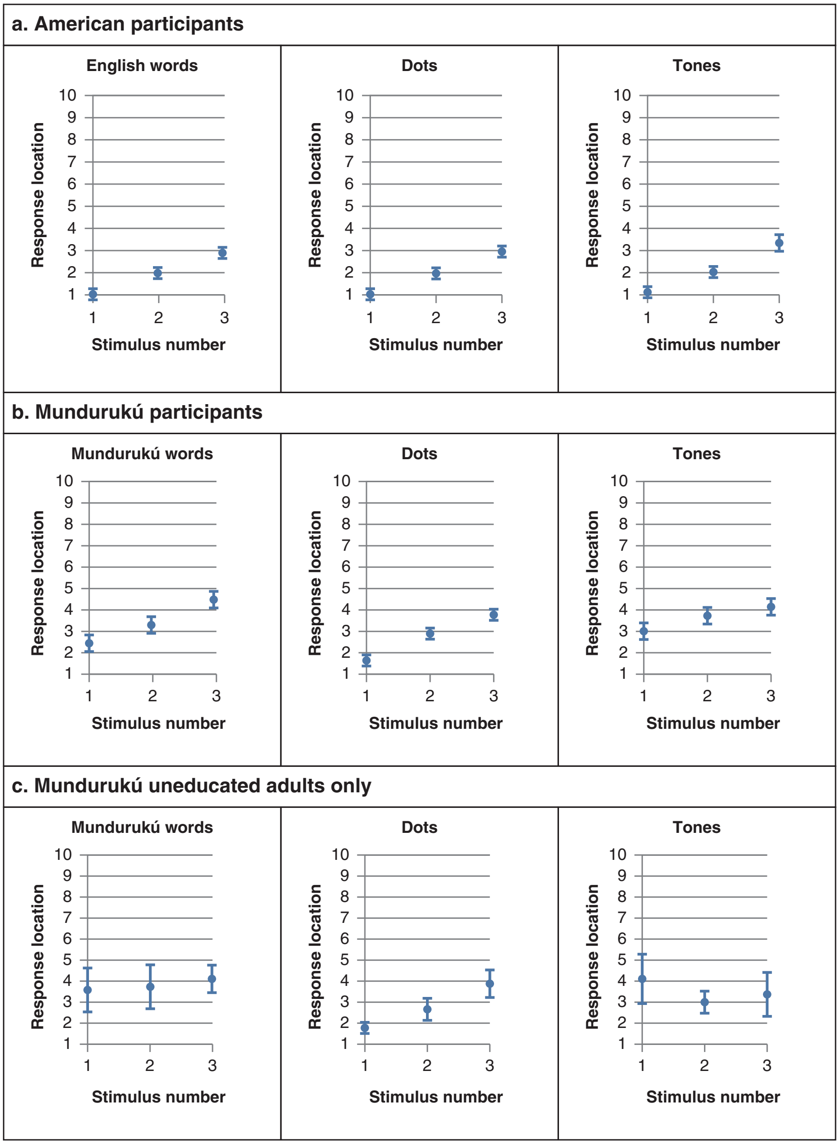

Figure 3.7 Detail from data reported in Dehaene et al. (2008a, 2008b) showing the mean response location on the line segment that participants picked as corresponding to the numerosities 1, 2, and 3—numbers for which Mundurukú speakers have a well-established lexicon. Data are mean ± standard error of the mean. The top row (a) shows values for 16 American participants; the middle row (b) shows values for 33 Mundurukú participants; and the bottom row (c) shows values for 7 Mundurukú uneducated adults.(*Cont.*)

Figure 3.7 *(Cont.)* If participants understand the task and spontaneously map numbers to the line, they should systematically map the lowest stimulus number ("1") with the response location "1"—that is, with the left end of the line segment presented to them. American participants (a) accurately and systematically mapped "stimulus number 1" with "response location 1." Mundurukú participants, however, systematically failed to establish this fundamental number–space mapping (b). Crucially, data reported exclusively in the corresponding "Supporting Online Material" (Dehaene et al., 2008b) show that the Mundurukú uneducated adults—the most relevant subgroup for testing the innate mental number line hypothesis—failed to do this in a more dramatic way (c). For words and tones, this group even failed to establish the fundamental property of order. These results suggest that the number-line intuition is not universal.

Source: From Núñez, 2011, replotted from Dehaene et al., 2008a (Figures 3.7a and 3.7b) and Dehaene et al., 2008b (Figure 3.7c), with permission from the American Association for the Advancement of Science.

cannot be interpreted as number-line mappings proper. Moreover, when their frequency is considerably high, they demand further analyses. Strikingly, in the Mundurukú study, 13 experimental runs out of 35 (37%) were labeled as bimodal (Dehaene et al., 2008b)—a very high percentage considering the claim that the intuition of the number-line mapping is *universally* spontaneous. Indeed, according to the universality claim, no runs *at all* should be expected to be bimodal. In the Mundurukú study, however, even if the universal hypothesis is considerably loosened to accept as many as 20% of the runs to be bimodal, the observed frequency of the reported bimodal responses is still statistically significant ($\chi^2 = 6.43$, df = 1, $p = 0.011$). This high proportion of bimodal responses is simply at odds with the conclusion that indigenous people without instruction spontaneously map numbers onto a line.

Furthermore, the high proportion of bimodal responses helps explain another important problem in the data: the reported Mundurukú "mean response locations" do not corroborate that the Amazonian participants did establish the lowest-number-to-endpoint mapping—an essential component of the number-to-space mapping as specified by the number-line task (Siegler & Booth, 2004; Siegler & Opfer, 2003). A detailed analysis of the data shows that the Mundurukú—especially the unschooled ones, which is the group that matters the most for testing universal claims—failed to establish the basic "numerical anchors" (Booth & Siegler, 2006). Figure 3.7 shows that while the mean response locations of control (American) participants accurately and systematically mapped "stimulus number 1" with "response location 1"—the left end point of the segment (Figure 3.7a), the values for the Mundurukú participants show that they routinely failed to do so (Figures 3.7b and 3.7c). This is even more striking as the failure involved the number and numerosities "one," "two," and "three," which are the only cases for which Mundurukú speakers systematically use specific lexical items to identify them (the only terms used in more than 70% of the cases in which the corresponding quantity was presented; Pica et al., 2004).

The left graph of Figure 3.7b shows that for word numerals in Mundurukú (symbolic stimuli), the mean response location for the smallest stimulus number "1" was approximately "2.5"—that is, 1.5 location units to the right of the left endpoint. Moreover, the standard error, being about 0.5, indicates that there is a substantial variability in the response, meaning that for some participants, the mapped location for stimulus number "1" was even farther away from the left endpoint of the segment. Similarly, the center and right graphs of Figure 3.7b show that for dots and tones (nonsymbolic stimuli), the fundamental mapping 1-to-left endpoint was not established either. The mean response location for "one tone" was nearly 2 location units away from the left endpoint— that is, a distance corresponding to nearly 22% of the extension of the segment. Most important, data reported only in the corresponding Supporting Online Material show that for the Mundurukú uneducated adults—the most relevant subgroup for testing the innateness hypothesis—the failure was even more telling. The left and right graphs in Figure 3.7c show that the mean mapped location for the word "one" and for "one tone," respectively, was roughly 3 location units away from the left endpoint (about 30% of the extension of the segment), with a very high standard error, indicating that for some participants, the mapped location was even farther away to the right. Moreover, the uneducated Mundurukú adults even failed, for these cases, to establish the fundamental property of *order*: the mean response location for stimuli numerosities "1," "2," and "3" are virtually the same, with "one tone" being even farther to the middle of the line segment than "two tones" and "three tones." (Note: In this study, the presented line segment had a set of 1 dot on the left and a set of 10 dots on the right constantly present on screen; Dehaene et al., 2008a, p. 1218. This may explain the presence of order for the responses in the "dots" condition [center of Figure 3.7c]. These responses may have been driven by a perceptual resemblance between the numerical stimulus [dots] and the presented segment [with dots at the endpoints], a situation that was not available for the "word" and "tone" conditions.)

Considering the high proportion of bimodal Mundurukú responses, the "mean" responses reported in this study are then largely based on artifactual values obtained from averaging a high proportion of endpoint responses. Therefore, they should not be considered as central tendency measurements characterizing actual locations on the extent of the line segment. Rather, they should be considered as relative proportions of left- to right-endpoint responses. In conclusion, the high bimodal Mundurukú responses, which are categorical responses that fail to reflect psychological distances with a metric, support, in fact, the opposite claim defended by the authors: the spontaneous intuition of mapping numbers to linear space (with a distance function) is not universal.

Number Concepts without Number Lines in the Yupno of Papua New Guinea

A recent study that used essentially the same methodology as the one in the Mundurukú study was conducted with the Yupno of the mountains of Papua New Guinea (Núñez et al., 2012). Unlike the Mundurukú, the Yupno have number concepts and a number lexicon beyond 20, and they have access to a creole (Tok Pisin) with an English-based number lexicon. Like other indigenous groups in Papua New Guinea (Saxe, 1981), the Yupno have a body-count system, which establishes a number-to-space mapping that exhibits (local) properties of order but that lacks a metric—that is, a distance function mentioned earlier in this chapter. Importantly, however, they lack tools and practices for precise space or time measurements (Wassmann & Dasen, 1994). Other groups in Papua New Guinea make use of simple measuring practices, such as using the arms for measuring the depth of string bags (Saxe & Moylan, 1982), but the Yupno do not exhibit such practices. The Yupno, then, provide an excellent cultural group for testing the possibility that number concepts may exist without measurement practices and without spontaneous number-to-space mapping intuitions.

As in the Mundurukú study, Núñez and colleagues (2012) used symbolic (words) and nonsymbolic stimuli (dots and tones) in the 1–10 range. Because one of the main goals of the study was to test the genuine spontaneity of number-line mappings, the researchers carefully scripted the task instructions with specific wordings and gestures intended to avoid any unwitting scaffolding. Moreover, the study considered two sets of instructions. One version (Type-1) which had two number-anchors (1 and 10) and static descriptions. A second version (Type-2)—designed to be more explanatory—had three number-anchors (1, 10, and 5) and linguistic expressions that used more dynamic descriptions. While Type-1 instructions followed the procedure of the Mundurukú study as closely as possible, Type-2 instructions were added to test whether an explanation making overt use of the extent of the path might prompt the number-line intuition. The study included unschooled Yupno participants and Yupno participants who had a very limited education. To make sure that Yupnos participating in the number-line task understood the 1–10 cardinal numerical lexicon, the researchers designed a simple screening procedure that explicitly tested their mastery of the cardinal number lexicon in the Yupno language. (With very rare exceptions, all participants passed the number screening procedure.)

After running the number-line task, the researchers found that unschooled participants, despite knowing the number terms, had significant difficulties with the training trials, failing to comprehend the fundamental endpoint anchoring required by the number-line task (see Núñez et al., 2012, for details). The proportions of failures were significantly higher than those observed for the schooled Yupno participants, who exhibited no endpoint-matching failures. Crucially, the Type-2 instructions were not helpful, despite involving dynamic language that explicitly showed the mapping of the number 5 onto a location on the path.

Most important, analyses conducted on blocks that had successful endpoint anchoring trials (with standard Type-1 instructions) showed that unschooled Yupnos produced a mapping that systematically ignored the extent of the path. In all three stimulus modalities, they exhibited a metric-free bicategorical mapping, where small numbers and numerosities (1, and sometimes 2 and 3) mapped onto the left endpoint and

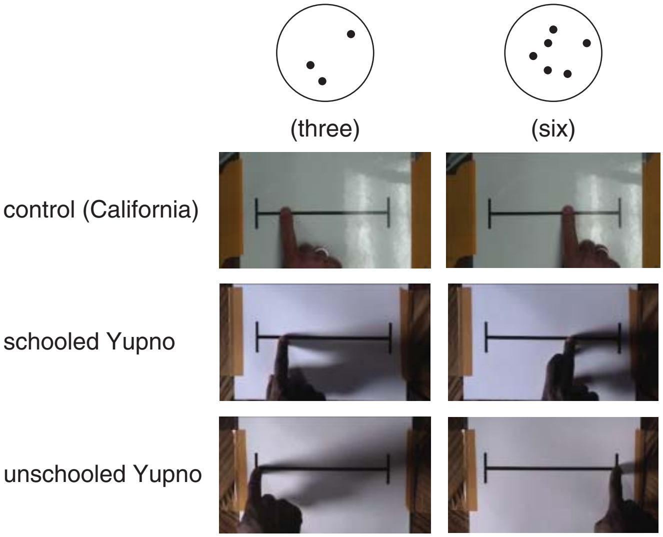

Figure 3.8 Pointing responses for stimulus numerosities 3 and 6 (dots) during the number-line task.
Source: From Núñez et al., 2012.

midsize and large numbers and numerosities onto the right endpoint (see Figure 3.8 for examples).

The bicategorical mappings produced by unschooled Yupnos— with no use of the extent of the segment when mapping intermediate numbers and numerosities—are in stark contrast to those of schooled Yupnos and California controls (Figure 3.9). The high proportions of endpoint responses by unschooled Yupno were extremely significant in all three stimulus modalities. And, while schooled Yupnos used the extent of the segment according to known standards (Dehaene et al., 2008a; Siegler & Booth, 2004), they did so not as smoothly as California controls, exhibiting a bias toward the endpoints. This schooled Yupno response pattern suggests an intermediate mastery of the number line that is culturally modulated, in which the primacy of the basic bicategorical intuition coexists with the learned distance function of the number-line mapping.

The Yupno bicategorical responses appear to be of the same type as the frequent "bimodal" responses mentioned (but ignored and left unanalyzed) in the Mundurukú study. The modes of the distributions are, in both cases, concentrated at the endpoints, with a significant

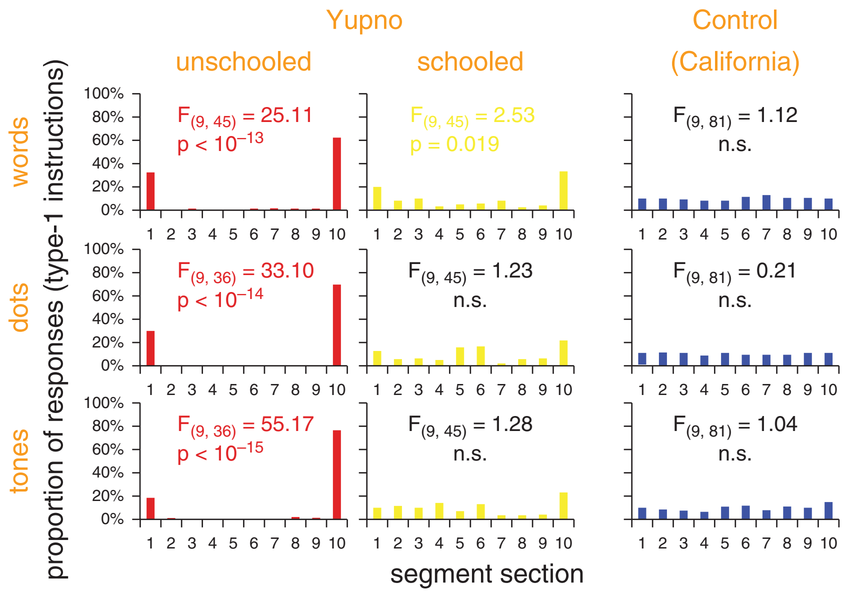

Figure 3.9 Responses to three different types of number stimuli on the number-line task. Graphs show proportions of responses on segment sections with corresponding repeated measures ANOVA statistics. In stark contrast with schooled Yupnos (middle column) and California controls (right column), unschooled Yupnos (left column) did not use the extent of the segment when mapping intermediate numbers and numerosities, producing instead a bicategorical mapping without a metric (a distance function). Schooled Yupnos (middle column) did use the line but tended to manifest a bias toward the endpoints.

Source: From Núñez et al., 2012.

amount of responses failing to use the extension of the line segment to map intermediate number and numerosity stimuli. The Mundurukú endpoint responses, although statistically significant, were not as marked as they were among the unschooled Yupno. But this difference could be explained by variations in the experimental procedures employed in the two studies. Mundurukú participants may have been helped by perceptual resemblance cues as they were presented, at all times and for all conditions, with an image of a line that had 1 dot and 10 dots located at the left and right end of the segment, respectively. The crucial point is that the significant bimodal Mundurukú responses are not simply oddities that can be ignored. Instead, they constitute evidence of a genuine phenomenon of lack of spontaneous number-line intuitions, a phenomenon that is manifested also in a completely unrelated cultural group—the Yupno.

The bicategorical pattern found in both the Mundurukú and Yupno groups could be interpreted as a manifestation of a more fundamental phenomenon described in developmental psychology. In the investigation of categorical and comparative interpretations by children, Smith, Rattermann, and Sera (1988) stated that "young children seem to interpret dimensional terms categorically even when the intended meaning, by adult standards, is clearly comparative" (p. 342). And in their research, they found that the categorical and comparative usages of young children (3-year-olds)—for example, high(er) and low(er)—are restricted to the corresponding poles, and it is only later that children via a richer use of comparatives understand quantitative dimensions as one encompassing system. Referring to earlier work on semantic incongruity effects in adults in which adults' reaction times suggest that they do not automatically process the logically necessary relation between opposing terms (Banks, 1977), Smith et al. conclude that "children's initial treatment of opposing terms as separate may reflect some fundamental fact about human cognition since the treatment appears both developmentally primitive and, in adults, computationally simple" (p. 356). It is possible then that in the Mundurukú and Yupno cultures, the pressure for dealing with quantitative dimensions as one encompassing system has not been as intense as it has in Western civilization, keeping the saliency of the poles much more prominent. And, in the case of the Yupno, this would be consistent with the fact that they do not have measuring practices of any sort and, importantly, their language does not have comparatives in the grammar (J. Slotta, personal communication, June 28, 2013) that might help solidify the concept of quantitative dimensions. Thus during the number-line task, when confronted to a notational device never seen before—the line segment—the unschooled participants from these cultural groups may have enacted this basic polar construal built around the (opposite) endpoints of the segment onto which small and big numbers were mapped.

The results from the Yupno and the Mundurukú studies, when taken together, demonstrate that the intuition of the number-to-line mapping as reflected by the number-line task is not universally spontaneous, and therefore, it is unlikely to be rooted directly in brain evolution. Although ubiquitous in the modern world, the number-to-line mapping seems to be learned through—and continually reinforced by—specific cultural practices. This conclusion is consistent with the findings in the study on number mappings with nonspatial reports discussed in the previous subsection (Núñez et al., 2011), showing that number-to-space mappings are not as fundamental as previously thought and that linear number-line mappings require cultural practices to be established. Moreover, these observations also seem to match the available records of the history of mathematics discussed earlier, which show no documentation of depictions of number lines proper prior to the 17th century (Núñez, 2009, 2011).

In sum, evidence from several sources—from developmental to cross-cultural—points to a basic human *association* between number and space. It is possible that this constitutes an early *precondition* for the establishment of number-line concepts. But the results analyzed in this section suggest that the emergence of the number line proper (i.e., with numbers mapped onto a unidimensional space with a metric—a distance function) requires considerable sociohistoric and cultural mediation, acting outside of natural selection in biological evolution. These basic *associations*, therefore, may be preconditions for the number line, but they are not precursors of it. Specific cultural practices, such as the use of measurement tools, notational and graphical devices, writing systems, and the implementation of systematic education, have served to establish, refine, and sustain the specificities of the number-line mapping. As recent work in the neuroscience of number cognition (Ansari, 2008) suggests, it is likely that, over the course of prolonged and systematic exposure to these cultural practices in educated individuals, brain areas such as the parietal lobes are recruited to support number-line representation and processing.

BIOCULTURAL ISSUES FOR CHILD PSYCHOLOGY AND DEVELOPMENTAL THEORY: IS SNOWBOARDING "HARDWIRED"? NO, IT IS NOT

The view that the number line is "hardwired" is attractive and convenient but oversimplistic and misleading. It contains teleological components that hide the sociotemporal complexity of human high-order cognition and its biocultural underpinnings. An analogy may help identify the teleological argumentation of the nativist position. In order to make the points more salient, rather than asking the question of the nature of mathematics and of related basic intuitions such as the number line, let us consider asking the question of the nature of snowboarding. Mathematics, like snowboarding, is observed only in humans. But if we think of a situation in which we would live in a global village where all (or most of the) humans we would normally encounter practiced snowboarding as the major form of locomotion, we would take snowboarding—like the number line—fully for granted and see it as a "natural" and "spontaneous" activity performed by (most) healthy individuals. In such a world, when scanning brains and studying neurological injuries of fellow snowboarders, we would be led to believe that there are "snowboarding areas" in the human brain, and we would see crawling infants as engaging in "protosnowboarding" and manifesting "early-snowboarding" capacities. Without a detailed, systematic, and in-depth investigation of the brains and behaviors of human ancestors' locomotion (even outside of our little snowy mountain) and of exotic no-snowboarding humans, we would be easily led to consider the basic components of snowboarding as "hardwired."

From the understanding of the real world we inhabit, however, we unmistakably know that snowboarding was not brought forth by biological evolution—snowboarding is not part of the human biological endowment. Clearly, in order to snowboard, we need a brain and make specific use of neural mechanisms shaped by evolution such as balance regulation, optic flow navigation, appropriate motor control dynamics, and so on. These mechanisms may constitute early preconditions for snowboarding, but in themselves, these mechanisms are not precursors of snowboarding and are not *about* snowboarding: they cannot explain the emergence of snowboarding proper. In addition to certain environmental settings foreign to human biological evolution (e.g., mountain slopes with snow), snowboarding requires crucial cultural mediation. For instance, snowboarders must be members of a culture that has already solved problems of thermic insulation of the human body such that they don't suffer from lethal hypothermia when practicing such forms of locomotion. They must also be a part of a culture that has invented sophisticated materials for optimal board sliding that do not exist naturally in the environment and lift gondolas such that they can optimize the use of energy allowing for repeated downhill slides, which avoids the tiring and time-consuming climbing back up the hill after each slide. Such technology enables fast and efficient improvement of snowboarding techniques that develop in the ontogeny of individuals and the learning of a very unnatural locomotion pattern, one that clearly did not evolve through biological evolution: locomotion with highly restricted lower-limb movement independence occurring in freezing conditions. And so the analogy can be fleshed out with any desired level of detail.

The moral is that humans may indeed have evolutionarily driven "hardwired" mechanisms for numerosity judgments (e.g., subitizing, large numerosity discrimination provided that they differ by specific ratios) and perception of stimulus intensity (e.g., Weber–Fechner law) that may constitute early preconditions for numerical and arithmetical abilities. But these mechanisms in themselves do not provide an explanation of the nature of arithmetical or mathematical entities proper, not even of fundamental ones such as the number line: basic numerosity discrimination (which is biologically endowed) is to number-to-space mappings (culturally shaped and learned by individuals) as crawling is (not to walking but) to snowboarding. Evolutionarily shaped mechanisms such as optic flow navigation and numerosity discrimination can be recruited for the consolidation of behaviors involving snowboarding and the number line, respectively, but in themselves, they do not tell us about the nature of snowboarding or the mental number line. It is then misleading to teleologically consider young infants engaging in crawling locomotion and manifesting large numerosity discrimination abilities as "early" (or "proto-") snowboarding or as "early" (or "proto-") arithmetic, respectively. Arithmetic and the mental number line (as the term suggests) is about numbers, and numbers, as such, are much more than numerosity judgments and perceptual discrimination capacities. They are sophisticated human symbolically sustained concepts that, while they build on biological resources and constraints, are culturally and historically mediated by language, external representations, writing and notation systems, and the need to solve specific societal problems (Núñez, 2009). None of these crucial components that make numbers and the number line possible is "hardwired," just as thermic insulation and gondola technology in the case of snowboarding is not "hardwired."

The claim that the mental number is hardwired can be tested diachronically and synchronically. If the mental number line is shared by all humans, we must observe manifestations of it (1) throughout the history of humankind and (2) in all cultures around the world today. But we don't. Archaeological and historical data—from Old Babylonia to 17th-century Europe—are simply at odds with the idea of a "hardwired" mental number line that would spontaneously manifest itself in all humans. And unschooled Mundurukú adults dramatically failed to map even the simplest numerosity patterns—one, two, and three—with a line segment, and a high proportion of them used only the segment's endpoints, failing to use the full extent of the response continuum. The same bicategorical response was observed in a completely unrelated culture on the other side of the globe, with the totality of the unschooled Yupno participants across all stimulus modalities. Establishing the amazingly efficient number-to-space mapping as the number line is an extraordinary fact in human history, which took centuries—if not millennia—of cultural inventions building on strongly constrained human cognitive capacities. The mental number-line phenomena we observe in people these days seem to be the manifestation of the psychological and neurological realization of today's well-established number line (and its related mental representations) that people acquire through cultural practices and educational exposure. The currently known behavioral and neuroimaging data, almost exclusively gathered with schooled participants from the industrialized world, actually reveal how the neural phenotype realizes the culturally created number line, rather than give support to the idea that all *Homo sapiens* map numbers to space in a number-line manner.

In this chapter I took on the question of how much of mathematics is "hardwired" (if any!). I addressed it primarily with the relatively simple case of the number line, to conclude that the spontaneous intuition of number-to-line mappings is not innate, not part of the human biological endowment, but rather it requires considerable cultural mediation and scaffolding for it to be brought to being. And what about more elaborated mathematical concepts? Are they "hardwired"? I leave it to the reader to come up with his or her own conclusion.

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