1. $B_0$ field from superconducting magnet
2. Gradient coils
3. Body RF transmit/receive
4. RF receive-only
5. Shim coils (in gradients)

$B_0 \rightarrow z$ (longitudinal)
$B_1 \rightarrow x, y$ (transverse)

Max gradient:
- $80 \text{ mT/m}$
- $200 \text{ Tm/sec}$

(1) $B_0$ field
(2) Body gradient coils
(3) RF transmit body coil
(4) RF receive-only head coils

$(1 \text{T} = 1,000 \text{ Gauss})$
Earth: $0.25 - 0.65 \text{ G}$
$25 - 65 \text{ kHz}$
$rac{y}{2\pi} = 42 \text{ MHz/T}$

RF transmitter (30 kW)
RF receiver
Circularly polarized $B_1$ field rotating to $B_0$ at Larmor freq

$B_1$ field is several orders of magnitude smaller than $B_0$.

Non-superconducting water-cooled, external shield

Superconducting coils in liquid helium (no power required after current injected to bring up field using induction).
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers).
- Moving charge creates magnetic field.
- Current loop from spinning charge (right-hand rule).
- N.B.: Classically this would cause EM radiation, spin-down.
- Stern-Gerlach experiment:
  - Pass silver atoms thru strong mag. field → split into just 2 beams.

### Microscopic Picture

- No strong magnetic field, \( B_0 = 0 \)
- Strong magnetic field, \( B_0 \)
- Strong \( B_0 \) plus oscillating \( B_1 \)

### Macroscopic Picture

- Bulk magnetization may depend on \( B_0 \)
- Precessing vectors are "bunched" at any one moment around circle.

**Precession**

- Distinguish precession (slow) from spin (fast).
- Treat classically, like spinning top.

\[
2\pi f = \frac{\hbar}{B_0} = \frac{q I B_0}{\hbar} \quad \text{Larmor freq.} \quad \text{rad/s} \quad \text{gyro-magnetic ratio}
\]

- Bulk equilibrium magnetization (parallel to \( B_0 \))

\[
M_z = |\vec{M}| = \frac{\gamma^2 \hbar^2}{4 KT} B_0 N
\]

where \( I = \pm \frac{1}{2} \)

\[
I(I+1) = \frac{1}{3}
\]

\[
\gamma = \text{gyromagnetic ratio}
\]

\[
h = \text{Planck's const.}
\]

- \( B_0 \) → i.e., \( M_z \) proportional to \( B_0 \) strength.
- \( N_0 \) → i.e., \( M_z \) proportional to number spins.
- \( K = \text{Boltzmann const.} \)
- \( T_0 = \text{abs. temperature sample} \)
**Bloch Equation**

- Time-dependent behavior of \( \vec{M} \) in the presence of an applied magnetic field (excitation + relaxation)

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{Y} - \frac{\vec{M} \times (\vec{B}_0)}{T_2} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{M_z - M_z^0}{T_1} \text{ unit vector z-dir.}
\]

- In the Larmor-rotating coordinate system, a tilt or a phase shift from a standard \( B_1 \) excitation is rotation around x-axis

- Longitudinal and transverse relaxations

\[
\frac{dM_z(t)}{dt} = - \frac{M_z(t) - M_z^0}{T_1}
\]

\[
\frac{dM_x(t)}{dt} = - \frac{M_x(t)}{T_2}
\]

- Solution to equations above: time-dependent free precession eg's

\[
M_z'(t) = M_z^0 \left( 1 - e^{-t/T_1} \right) + M_z'(0)e^{-t/T_1}
\]

\[
M_x'(t) = M_x'(0)e^{-t/T_2}
\]
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers
  - Generalizes to any D

- Multiple ways to multiply vectors: here are 3

**Dot product**

- Inner product
- "Scaled projection onto"

\[ \vec{c} = \vec{a} \cdot \vec{b} = [a_x b_x, a_y b_y, a_z b_z] \]

\[ p = \|\vec{a}\| \cos \theta \quad c = p \|\vec{b}\| \quad c = \|\vec{a}\| \|\vec{b}\| \cos \theta \]

**Cross product**

- Outer product
- Can be generalized: see "geometric algebra"

\[ \vec{c} = \vec{a} \times \vec{b} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]

- Right-hand rule: curl fingers from \( \vec{a} \) to \( \vec{b} \); thumb is \( \vec{c} \)
- Unique orthogonal specific to 3D

- Skew-symmetric: \( \vec{a}^T = -\vec{a} \)

**Complex multiply**

- See also quaternions, geometric algebra generalization

\[ \vec{c} = \vec{a} \cdot \vec{b} = \begin{bmatrix} b_x & -b_y \\ b_y & b_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]

- Angles add
- Magnitudes multiply
- Specific to 2D

- \( \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \) 
- Like real nums

- L.B. equals: \( \vec{a} \cdot \vec{a} \)

- Geometric algebra: bivector plane area
**Effects of \( \vec{M}, \vec{B}, \) and \( \Theta \) on PrecessionFreq.**

\[ \frac{d\vec{M}}{dt} = \vec{M} \times \vec{B} \]

- **Cross Prod. Properties Review:**
  \[ \left\| \frac{d\vec{M}}{dt} \right\| = \left\| \vec{M} \right\| \cdot \left\| \vec{B} \right\| \cdot \sin \Theta \]

- **Starting Condition:**
  \( \Rightarrow \) now see effects of changing \( \vec{M}, \vec{B}, \Theta \)

- **Change \( \vec{M} \) Length:**
  \( \Rightarrow \) \( \frac{d\vec{M}}{dt} \) proportionally larger, so cancel effect of larger \( \vec{M} \)
  \( \Rightarrow \) same precession freq. as starting cond.

- **Change \( \Theta \) Between \( \vec{M} \) and \( \vec{B} \):**
  \( \Rightarrow \) \( \frac{d\vec{M}}{dt} \) goes up (then down) as \( \sin \Theta \)
  \( \Rightarrow \) but circumference also goes up as \( \sin \Theta \), cancelling again
  \( \Rightarrow \) same precession freq.

- **Change \( \vec{B} \) Length:**
  \( \Rightarrow \) \( \frac{d\vec{M}}{dt} \) goes up, proportional to \( \vec{B} \)
  \( \Rightarrow \) but circumference is same at starting cond.
  \( \Rightarrow \) increased precession freq. \( (\omega = \gamma \Omega) \)
**Simple Matrix Operations**

**Basic Idea**
- a matrix \([\text{rotates/scal}es]\) a vector
  \[ \mathbf{b} = \mathbf{M} \hat{\mathbf{a}} \]

**3D Example**
\[
\begin{bmatrix}
 b_x \\
 b_y \\
 b_z
\end{bmatrix}
= \begin{bmatrix}
 M_{11} & M_{12} & M_{13} \\
 M_{21} & M_{22} & M_{23} \\
 M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
 a_x \\
 a_y \\
 a_z
\end{bmatrix}
\]

**Add Translate (after rotate/scale)**
- commonly used "hack" for aligning verts
- a 4D matrix \([\text{rotates/scal}es/\text{translates}]\) a 3D vector
  \[ \begin{bmatrix}
 b_x \\
 b_y \\
 b_z \\
 b_w
\end{bmatrix}
= \begin{bmatrix}
 M_{11} & M_{12} & M_{13} & M_{14} \\
 M_{21} & M_{22} & M_{23} & M_{24} \\
 M_{31} & M_{32} & M_{33} & M_{34} \\
 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 a_x \\
 a_y \\
 a_z \\
 a_w
\end{bmatrix}
\]

**Translate?**
- N.B.: Have to keep track of order!!
  - rotate/scale then trans ≠ trans, then rot/scal
  - change rot component: untranslate, rot, retranslate

**3 Special Cases (3D):** rotate around each major axis without changing length (Scale = 1.0)

- **Rotated around x-axis:**
  \[
  \mathbf{R}_x(\theta) = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \theta & -\sin \theta \\
  0 & \sin \theta & \cos \theta
\end{bmatrix}
  \]
  e.g., 90° flip

- **Rotated around y-axis:**
  \[
  \mathbf{R}_y(\theta) = \begin{bmatrix}
  \cos \theta & 0 & \sin \theta \\
  0 & 1 & 0 \\
  -\sin \theta & 0 & \cos \theta
\end{bmatrix}
  \]
  e.g., 180° flip to avoid add 180° phase after 90° flip on x'

- **Rotated around z-axis:**
  \[
  \mathbf{R}_z(\theta) = \begin{bmatrix}
  \cos \theta & -\sin \theta & 0 \\
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
  \]
  e.g., precession with BP along z'

**General Case**
- rotate around general \(z'\)-axis
  \[
  \mathbf{R}_{z'}(\phi) = \mathbf{R}_z(-\theta) \mathbf{R}_y(-\phi) \mathbf{R}_z(\theta) \mathbf{R}_y(\phi) \mathbf{R}_z(\theta)
  \]
  (quaternions are more efficient)
**Solutions to Simple Differential Eq.**

**Diff. Eq.:**
\[
\frac{dM_{xy}(t)}{dt} = -\frac{M_{xy}(t)}{T_2}
\]

**Solution:**
\[
M_{xy}(t) = M_{xy}(0) \cdot e^{-\frac{t}{T_2}}
\]

**Goal:**
1) Find \( \varphi \) whose derivative satisfies diff. eq.
2) Also find soln. (one of many) that passes thru init condition

\( \rightarrow \) since own diff. eq. is:
- derivative of funct. = const. same funct.
- try exponential, since derivative \((e^x)' = e^x\)

**First Solution:**
\[
M(t) = e^{-\frac{t}{T_2}}
\]

**Derivative to check:**
\[
M'(t) = -\frac{1}{T_2} \cdot M(t)
\]

**Initial Condition:**
\[
M(t) = M_{xy}(0) \cdot e^{-\frac{t}{T_2}}
\]

**Magneticization immediately after RF (B1) ends:**
\[
M_{xy}(t) = M_{xy}(0) \cdot e^{-\frac{t}{T_2}}
\]

**N.B.:**
- This function is the "unknown" like the \( x \) in \( x + 1 = 3 \)
- Family of solutions
- Initial condition

**Information added to soln. (not from diff. eq.):**
\[
M'(t) = \frac{M_{xy}(0)}{T_2} \cdot e^{-\frac{t}{T_2}}
\]
**Bloch Eq. - Matrix Version**

Differential Eq.:

\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}_\phi
\]

Solution:

\[
\mathbf{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0_t) \\ M_y(0_t) \\ M_z(0_t) \end{bmatrix} = \mathbf{R}_z(\omega t) \mathbf{M}(0_t)
\]

Include Relaxation

Differential Eq.:

\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}_\phi - \frac{M_x I + M_y J}{T_2} - \frac{(M_z - M_z^*) K}{T_1}
\]

Solution:

\[
\mathbf{M}(t) = \begin{bmatrix} e^{\frac{t}{T_2}} & 0 & 0 \\ 0 & e^{\frac{t}{T_2}} & 0 \\ 0 & 0 & e^{\frac{t}{T_1}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0_t) \\ M_y(0_t) \\ M_z(0_t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0_t)(-e^{-\frac{t}{T_1}}) \end{bmatrix}
\]
**Excitation in the Rotating Frame**

- **Original Bloch eq.**
  - in laboratory frame
  - \( \frac{d\vec{M}_{\text{lab}}}{dt} = \vec{M} \times \vec{B} \)

- **Add on-resonance B1**
  - to tip

- **Substitution to convert to the rotating frame**

- **Substitution after any off-resonance appears as residual \( B_0 \) (B2)**
  - See off-res notes page

- **Rotating frame < on-resonance**
  - basic excite, B1x-only
  - \( \vec{M} = \vec{R}_z(w_t) \cdot \vec{M}_{\text{rot}} \)
  - \( \vec{B} = \vec{R}_z(w_t) \cdot \vec{B}_{\text{rot}} \)

- **Rotating frame < off-resonance**
  - general, B1x-only incl gradients

- **Rotating frame < small tip approx.**

- **Matrix version**

- **Matrix multiply**
  - \( \vec{M}_{\text{lab}} = \vec{M} \times \vec{B} \)

- **Small tip**
  - \( M_z = 0 \)
**Bloch Eq. Summary**

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{\mathbf{B}} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_0) \hat{k}}{T_1}
\]

(Lab frame)

(full lab frame picture is complex:
- 3 component \( \frac{d\vec{M}}{dt} \) update vector
- Larmor freq. component 7-9 orders magnitude larger than \( T_2, T_1 \) decay
- \( \vec{B}_1 \) is also rapidly wiggling)

Conceptual simplification in 4 stages:

1) **Lab frame**
   - just precession
   - \( \vec{M} \) stopped
   - that is, \( B_0 = 0 \)

2) **Rotating frame**
   - \( \vec{M} \) stopped
   - but \( \vec{M} \times \vec{\mathbf{B}} \) still works!!
   - "precess" around \( \vec{B}_1 \) axis

3) **Add \( \vec{B}_1 \)**
   - \( \vec{B}_1 \) also stopped!

4) **Off-resonance**
   - Slow precess, now around tilted \( \vec{B}_0 \)
   - Tilted plane
   - Apparent \( B_z \) comp. from residual precess. around \( z \) from off-resonance
RF Field Polarization

- Polarization (change of direction)
- Linearly polarized field
  \[ \vec{B}_1(t) = B_1 \cdot \cos \omega t \hat{\mathbf{z}} \]
  Magnet strength: \{1, 1\} \cdot 1
- N.B.: \( \vec{B}_1 \) adds to much larger \( \vec{B}_0 \)

\[ \vec{B}_0 + \vec{B}_1 \]

- Circularly polarized field (quadrature)
  \[ \vec{B}^{\text{circ}}_1(t) = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \]
  \[ = B_1 \cdot e^{-i \omega t} \]

- In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF spin

Typical 90° flip
(around x-axis)

Typical 180° flip
(around opposite y-axis)

180° flip regk ~6x power of 90°
**SIGNAL EQUATION**

\[ \Phi(t) = \int_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r},t) \, d\mathbf{r} \]

magnetic flux thru coil (integral of mag. field perpendicular to area)

- time deriv. inside - use Bloch precess. soln.
- deriv. cos \( \rightarrow \) sin
- deriv. sin \( \rightarrow \) cos

\[ V(t) = - \frac{d}{dt} \Phi(t) = - \frac{2}{\mu_0} \int_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r},t) \, d\mathbf{r} \]

Faraday law & Induction

- evaluate using free precession eqn. (solution to Bloch) ignoring relaxation
- rewrite w/ complex notation in time-dependence from lab frame Bloch

\[ \Phi(t) = \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r},t) \]

- ignore change in z-comp. \( \mathbf{M} \) because so slow \( \rightarrow \) i.e., we only see \( M_{xy} \), not \( M_z \)
- substitute \( \mathbf{M}(t) \) with lab frame \( M_{xy}(t) = M_{xy}(0) e^{-i wt} e^{-i wt} \)

- simplify:
  1) ignore decay (assume this \( t=0 \))
  2) assume phase-sensitive detection

\[ S(t) = \int_{\text{obj}} B_{xy} (\mathbf{r}) M_{xy}(0) e^{-i \omega t} \, d\mathbf{r} \]

\[ S(t) = \int_{\text{obj}} M_{xy}(\mathbf{r},0) e^{-i \omega t} \, d\mathbf{r} \]

i.e., at a single time point, RF signal is vector sum across object of local transverse magnetization vectors

\[ \omega t = \text{radians/sec} \]

Spatial phase in rotating frame

\[ \text{standard signal expression} \]

\[ w_t = \frac{\text{radians}}{\text{sec}} = \frac{\text{radians}}{\text{sec}} \]

\[ \phi = \omega \text{w} t \]

getting difference converts lab \( \rightarrow \) rotating frame
PHASE-SENSITIVE DETECTION

\[ V(t) \rightarrow \text{multiply} \rightarrow \text{Low-Pass Filter} \rightarrow S(t) \]

- method for moving very high frequency Larmor oscillations down to tractable frequency range

\[ \text{reference signal (123 MHz)} \]

\[ \text{demodulated signal} \propto \text{RF coil signal} \cdot \text{reference (transmitter)} \]

\[ \propto \sin[(\omega_0 + 2\omega_r)t] \cdot \sin[\omega_r t] \]

\[ \frac{1}{2} [\cos \omega_0 t - \cos (2\omega_0 + 2\omega_r)t] \]

\[ \text{filter this one out with low pass filter} \]

**One freq - freq domain**

\[ \text{signal} \rightarrow \omega \rightarrow \omega_0 \rightarrow \omega_0 + 2\omega_r \rightarrow \omega 
\]

\[ \text{reference} \rightarrow \omega_0 \rightarrow 2\omega_0 + 2\omega_r \rightarrow \omega_0 
\]

\[ \text{demodulated} \rightarrow 2\omega_0 + 2\omega_r \rightarrow \omega_0 
\]

\[ \text{after filter} \rightarrow \text{(rotating frame)} \]

**Chirp - time domain**

\[ \text{chirp} \rightarrow \text{center} \rightarrow \text{demodulated} \rightarrow \text{lo freq signal} \rightarrow \text{filter} \rightarrow \text{phase!} \]

- two signals are made from a single receiving RF coil
- a quadrature coil can be treated the same way (OK to combine after adding \(7\pi/2\) phase, then PSD)
- quadrature coil has better S/N since noise in each part is uncorrelated (\(\sqrt{2}\) better)

\[ \tilde{S}(t)_{\text{complex}} = M_{xy} e^{-i \omega_r t} \]
**FID - FREE INDUCTION DECAY, T2**

- Signal (FID) resulting from RF pulse w/ angle $\alpha$

\[
\vec{s}(t) = \sin \alpha \int_{-\infty}^{\infty} \rho(w) \cdot e^{-t/T_2(w)} \cdot e^{-i\omega t} \cdot dw
\]

- An example spectral density ("Lorentzian inhomogeneity")

\[
\rho(w) = \frac{M_0^2}{\left(\frac{\gamma AB\phi}{\Delta \omega}\right)^2 + \left(w - w_0\right)^2}
\]

- Combine $T_2 + \text{static terms}$

\[
\vec{s}(t) = \frac{\pi \cdot M_0^2 \cdot \gamma \Delta B\phi \cdot \sin \alpha \cdot e^{-t/T_2} \cdot e^{-i\omega t}}{\frac{c^2}{c^2 + w^2}}
\]

- $T_{2*}$ overall decay rate including inhomogeneous $B\phi$

\[
\frac{1}{T_{2*}} = \frac{1}{T_2} + \frac{1}{T_2'}
\]

- $\rho(w)$ not Lorentzian!
**ECHOES - spin echo**

- Just after 90° x' pulse, \( f_{lo} + f_{hi} \) have same phase.
- Relaxation + phase dispersion of \( f_{lo} + f_{hi} \) (both from \( B>B_{0} \)).
- Remember brief RF just tips vectors while retaining length.
- Relaxation includes tips and shrinks (\( M_{r} \)) and grows (\( M_{g} \), echo).
- 180° x' pulse works, too, but echo will have +\( \pi \) phase (left side in Figs above).
- Echo generated even if second pulse not 180° (see next).

**Rotating Coords**

- N.B.: Bloch eqs in one voxel.
- Rotate 180° around y'.
- Arrow below plane (from T2).
- Echo caused by re-phasing of \( f_{lo} + f_{hi} \) (w/ decay due to T2).

**Echo Decays (and echo growth/decay)**

- FID decay (and echo growth/decay) described by T2* from inhomogeneity.
- Reduction in height of echo compared to initial described by T2, echo fixes 'star'.

**FID - receive signal ampl.**

- Just suggests (really about 1,000 cycles here).
- Hidden, dephased transverse mag. here.
- N.B.: would be complete disk above.
**ECHOES — spin echo**

\[ \alpha_1 - \tau - \alpha_2 - \tau \] (both pulses along \( y' \) for simplicity)

---

**Effect of \( \alpha \) \( y \) Pulse**

\[
\begin{align*}
M_x' &\rightarrow M_x' \cos \alpha - M_z' \sin \alpha \\
M_y' &\rightarrow M_y' \\
M_z' &\rightarrow M_x' \sin \alpha - M_z' \cos \alpha \\
&\Rightarrow \text{(etc for \( \alpha_1, \alpha_2 \))}
\end{align*}
\]

---

**General Transform (operations)**

**Effect of \( \tau \) Delay**

\[
\begin{align*}
M_x' &\rightarrow (M_x' \cos \omega \tau + M_y' \sin \omega \tau) e^{-\gamma/2} \\
M_y' &\rightarrow (-M_x' \sin \omega \tau + M_y' \cos \omega \tau) e^{-\gamma/2} \\
M_z' &\rightarrow M_z'(1 - e^{-\gamma/2}) + M_z' e^{-\gamma/2}
\end{align*}
\]

---

**Immediately After \( \alpha_1 \) Pulse**

\[
\begin{align*}
M_x'(w, t_0) &= -M_z'(w) \sin \alpha_1 \\
M_y'(w, t_0) &= 0 \\
M_z'(w, t_0) &= M_z'(w) \cos \alpha_1
\end{align*}
\]

for one isochromat of freq. \( w \)

---

**After \( \tau \) Delay**

\[
\begin{align*}
M_x'(w, \tau) &= -M_z'(w) \sin \alpha_1 \cos \omega \tau e^{-\gamma/2} \\
M_y'(w, \tau) &= M_z'(w) \sin \alpha_1 \sin \omega \tau e^{-\gamma/2} \\
M_z'(w, \tau) &= M_z'(w) \left[ 1 - (1 - \cos \alpha_1) e^{-\gamma/2} \right]
\end{align*}
\]

---

**Immediately After \( \alpha_2 \) Pulse (no effect on \( M_y \); rewrite \( \alpha \) and \( y \) eqs.)**

---

**Time Dependent**

Free precession around \( z' \) (rewrite \( M_x', y, \tau + \))

---

**For a Large Num of Freq's:**

\[
\begin{align*}
[M_x', y, (w, \tau +)] &= M_x', y, (w, \tau) e^{-(t-\tau)T_2} e^{-i\omega(t-\tau)} \\
&= M_z'(w) \sin \alpha_1 \sin \omega \frac{\alpha_2}{2} e^{-T_2 e^{-i\omega(t-\tau)}} \\
&- M_z'(w) \sin \omega \frac{\alpha_2}{2} e^{-T_2 e^{-i\omega(t-\tau)}} \\
&- M_z'(w) \left[ 1 - (1 - \cos \alpha_1) e^{-\gamma/2} \right] \sin \omega \frac{\alpha_2}{2} e^{-T_2 e^{-i\omega(t-\tau)}}
\end{align*}
\]

---

**Echo Signal**

\[
\begin{align*}
S(t) &= \sin \alpha_1 \sin \omega \frac{\alpha_2}{2} \int_0^\infty \rho(w) e^{-t/T_2} e^{-i\omega(t-T_2)} \, dw \\
A_E &= \sin \alpha_1 \sin \omega \frac{\alpha_2}{2} \int_0^\infty \rho(w) e^{-T_2/T_2} \, dw = M_z'(w) \sin \alpha_1 \sin \omega \frac{\alpha_2}{2} e^{-T_2/T_2}
\end{align*}
\]

---

**Peak Amplitude**

\[
\begin{align*}
90^\circ, -\tau - 90^\circ \quad S_1(t) &= \frac{1}{2} \int_0^\infty \rho(w) e^{-t/T_2} e^{-i\omega(t-T_2)} \, dw \\
90^\circ, -\tau - 180^\circ \quad S_2(t) &= \frac{1}{2} \int_0^\infty \rho(w) e^{-t/T_2} e^{-i\omega(t-T_2)} \, dw \\
90^\circ, -\tau - 180^\circ \quad S_3(t) &= \text{no } \frac{1}{2} \text{ factor} \\
90^\circ, -\tau - 180^\circ \quad S_4(t) &= \text{multiply by } i \rightarrow \text{add } \frac{1}{2} \text{ phase}
\end{align*}
\]

---

\( \alpha_1 = 90^\circ \quad \alpha_2 = 90^\circ \)

\( e.g., \text{special case} \)

---

\( 2\gamma \)

---

**Echo Amplitude, ignoring freq. dependence of \( T_2 \)**

etc for \( A_E \) ... like ↑
ECHO TRAINS — spin-echo trains

- It's (too) easy to make echoes...

RF transmit

$T_1$  
$T_2$ (TM: time mixing) 

FID$_1$  
FID$_2$  
FID$_3$

$T_1$  
$T_2$

RF receive

$\alpha_1$  
$\alpha_2$  
$\alpha_3$

- A useful multi-echo sequence (CPMG) is a $90^\circ$ followed by $180^\circ$ at $2T$ spacing

RF transmit

$90^\circ$  
$180^\circ$  
$180^\circ$

RF receive

$e^{-t/T_1}$  
$e^{-t/T_2}$

- Typically, $90^\circ$ and $180^\circ$ applied in different axes ($x'$, then $y'$, $y''$,...), which reduces phase errors due to imperfect $180^\circ$ pulses (since slightly-off rotation around $y'$ affects phase less)

$E_n = \frac{3^{(n-1)} - 1}{2}$

Echoes after end of $n$th pulse

3 RFs $\rightarrow$ 4 echoes (here)
6 RFs $\rightarrow$ 121 echoes (!)

Secondary echo: SE$_{1,2}$ acts like RF pulse
$
\alpha_3$ makes an echo from it

Stimulated echo: combined effect of 3

$\alpha_1$: $M_L \rightarrow M_T$

$\alpha_2$: Leftover $M_T$ flipped to $M_L$ (saved)

$\alpha_3$: Flip saved $M_L \rightarrow M_T$ which can then begin to cancel delays (after being held in limbo between $180^\circ$ FID$_2$ and FID$_3$); acts like 2-pulse echo
EXTENDED PHASE GRAPHS

- Using full Bloch eq. solutions is tedious 😒
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize $90^\circ, 180^\circ$)
- Problem #1: $\alpha$ pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into $M_L$
- Problem #2: Third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

$\phi$ rule for effect of $\alpha$
RF pulse on transverse mag

$\phi$ rule for effect of $\alpha$
RF pulse on longitudinal mag

→ Echo when phase path crosses zero
3 - Pulse Echo Amplitudes

- Assume $M_z^0 = 1$

RF transmit:

RF receive:

<table>
<thead>
<tr>
<th>Echo</th>
<th>Time</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SE_{1,2}$ ($t = 2\tau_1$)</td>
<td>$\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2\tau_1/T_2}$</td>
<td></td>
</tr>
<tr>
<td>$2^o$ ($&quot;secondary&quot;$) ($t = 2T_2$) ($t = 2\tau_1 + 2T_3$)</td>
<td>$-\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2\tau_3/T_2}$</td>
<td></td>
</tr>
<tr>
<td>STE ($&quot;stimulated&quot;$) ($t = 2\tau_1 + T_2$)</td>
<td>$\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-\tau_3/T_2} e^{-2T_1/T_2}$</td>
<td></td>
</tr>
<tr>
<td>$SE_{2,3}$ ($t = \tau_1 + 2T_2$)</td>
<td>$[1 - (1 - \cos \alpha_i) e^{-\tau_1/T_2}] \sin \alpha_2 \sin^2 \frac{\alpha_3}{2} e^{-(\tau_1 + 2T_2)/T_2}$</td>
<td></td>
</tr>
<tr>
<td>$SE_{1,3}$ ($t = 2(\tau_1 + T_2)$)</td>
<td>$\sin \alpha_1 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2(\tau_1 + T_2)/T_2}$</td>
<td></td>
</tr>
</tbody>
</table>

- $T_1$-dependence in STE (but also $SE_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse
**HYPER ECHOES**

1. **3 solid lines 1 dashed line**
   - This is a spin echo!

(N.B.: coord not to put z horiz vs. Bloch notes)

2. **$\alpha_x \rightarrow 180^\circ, -\alpha_x \equiv 180^\circ$**
   - Normalize $\vec{M}$ amplitude $\rightarrow 1$
   - Sphere surface defines 2D space for $\vec{M}$ moved by:
     1) Vector rotation of $\vec{M}$ around tilted axis in transverse x-y plane by RF with flip, $\alpha_x$, and phase, $\phi = P(\alpha_x, \phi)$
     2) Rotation around z by phase evolution due to freq. offset, $\omega$ ($\omega_0$ offset) and time, $t = P(\omega, t)$
   - Three symmetries:
     - Solid lines: phase evolve or RF flip or RF
     - Dashed lines: just $180^\circ$ again
   - By combining long sequences observing these symmetries, can generate a strong echo even w/ many inserted $\alpha_x$-pulses in between

**Practical use**

- Multi-echo example
- Can also use to prepare, then separate read-out
- Practical prob: $180^\circ$ pulses deposit a lot of RF (6x 90°) → prob at high fields
- By arranging to get big echo in middle of k-space can get by with much less RF power

- Hennig & Scheffler (2001)
**Gradient Echoes** - T2*, GE chains

- Initial negative gradient dephases spins
- After $t=T$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities
  
  So echo amplitude is
  
  \[ A_E = e^{-t/T_2^*} \]

- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay

\[ \frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}} \]

- Key difference between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not canceled
  
  \[ \Rightarrow \text{hence, echoes are } T_2^* \text{-weighted, not } T_2 \text{-weighted } \Rightarrow \text{more susceptible to inhomogeneities} \]

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get

- EPI hardware \[ \Rightarrow \text{64 echoes} \]
IMAGE CONTRAST

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

RF

\[ \text{M}_2^0 \text{ longitudinal magnetization} \]

\[ \text{M}_2^0 (1 - e^{-TR/T1}) + [\text{ignored}] \]

- "steady state" after here

- Simple saturation/recovery w/ no echo

- Initial conditions:
  \[ \text{M}_2 \text{ before first pulse} = M_2^0 \]
  \[ \text{M}_2^0 = 0 \text{ immed. after first pulse} \text{ (i.e., 90° pulse)} \]

- From Bloch eq, \( M_z \) just before second pulse:
  \[ M_z^{(n+1)}(O) = M_z^{(0)} (1 - e^{-TR/T1}) + M_z^{(n)}(O) e^{-TR/T1} \]

- Given:
  1. 90° pulse
  2. no \( M_{xy} \text{ left} \)

  \[ \text{pure tip: } M_{xy} = M_z \]

- Tip existing mag
  \[ M_z^{(n)}(O) = M_x'^{(n)}(O) = M_z^0 (1 - e^{-TR/T1}) \]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[ I(r) = C \rho(r) \left( 1 - e^{-TR/T1(r)} \right) \]

Assume this is 0 because we assume \( M_{xy} \text{ (transverse) completely decayed so that a 90° pulse doesn't generate any initial longitudinal既然已知} \]

\[ \text{M}_z \text{ not fully decayed} \]

\[ 90° \text{ M}_z \]
IMAGE CONTRAST

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
  - because they are brighter than all the rest
  - because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
  (e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  - for typical 1-2 sec TR images reached after 8 images
IMAGE CONTRAST

1R (still just saturation-recovery — no echo)

- inversion recovery w/ no echo

RF

- 180° deg pulse reverses longitudinal magnetization
  \[ M_z' = -M_z \]

- recovery to end of first TI from long. part of Bloch eq.
  \[ M_z' = M_z \left(1 - 2e^{-\frac{TI}{T_1}}\right) \rightarrow \text{flipped into transverse by second pulse (180°)} \]

- longitudinal then regrows from zero from first Bloch term
  \[ M_z' = M_z \left(1 - e^{-\frac{(TR-TI)}{T_1}}\right) \]

- after second 180°, just change sign again
  \[ M_z' = -M_z \left(1 - e^{-\frac{(TR-TI)}{T_1}}\right) \]

- apply relaxation eq. again

\[ M_z' = M_z \left(1 - e^{-\frac{TI}{T_1}}\right) - M_z \left(1 - e^{-\frac{(TR-TI)}{T_1}}\right) e^{-\frac{TI}{T_1}} \]

\[ M_z' = M_z \left(1 - 2e^{-\frac{TI}{T_1}} + e^{-\frac{TR}{T_1}}\right) \]

\[ \rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
**IMAGE CONTRAST**

SE, IR-SE

- Steady state mag (2nd TR) just before 90°
  
  \[ M_z(0) = M_z^0(1 - 2e^{-(TR-TE/2)/T1} + e^{-TR/T2}) \]

- The echo signal (\(M_z^e\)) unlike in simple saturation-recovery FID
  
  Has an additional delay before it is recorded, so we have to take account of transverse mag relaxation
  
  \[ A_E = M_z^0(1 - 2e^{-(TR-TE/2)/T1} + e^{-TR/T2}) e^{-TE/T2} \]

- If we assume TE much less than TR, then we can simplify:

  \[ A_E = M_z^0(1 - e^{-TR/T1}) e^{-TE/T2} \]

- Similar equation for SE-IR

  \[ A_E = M_z^0(1 - 2e^{-TI/T1} + e^{-TR/T2}) e^{-TE/T2} \]
**IMAGE CONTRAST**

GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch eq. again
- Assume $M_{x'y'}^{(n)}(O_-) = 0$ -> transverse dephased before next pulse

$$M_{x}^{(n)}(O_-) = M_z^{0}(1 - e^{-TR/T_2}) + M_{x}^{(n-1)}(O_+)^e^{-TR/T_1}$$

- Assume we have a small tip angle:
  $$M_x^{0} \cos \alpha \Rightarrow M_{x}^{(n)}(O_+) = M_{x}^{(n)}(O_-) \cos \alpha$$

$$M_{x}^{(n)}(O_-) = M_z^{0}(1 - e^{-TR/T_1}) + M_{x}^{(n-1)}(O_-)^e^{-TR/T_1}$$

- Assume we are in dynamic equilibrium:
  $$M_{x}^{(n)}(O_-) = M_x^{(n-1)}(O_-) = M_x^{ss}(O_-)$$

**Prepulse**

$$M_{x}^{ss}(O_-) = \frac{M_z^{0}(1 - e^{-TR/T_1})}{1 - \cos \alpha e^{-TR/T_1}}$$

**Post-pulse**

$$M_{x'y'}^{ss}(t) = \frac{M_z^{0}(1 - e^{-TR/T_1}) \cdot \sin \alpha e^{-T_2*}}{1 - \cos \alpha e^{-TR/T_1}}$$

**Gradient echo amplitude**

$$A_E = \frac{M_z^{0}(1 - e^{-TR/T_1}) \sin \alpha e^{-TE/T_2*}}{1 - \cos \alpha e^{-TR/T_1}}$$

TI contrast mainly depends on flip angle, not TR $\rightarrow \cos 90^\circ = 1$ eliminates TI weight since denominator numerator.
Contrast - 4b

**IMAGE CONTRAST**

MDEPT / 3D FLASH

\[ \text{RF}_{\text{out}} \]

\[ G_z \]

\[ G_y \]

\[ G_x \]

\[ \text{RF}_{\text{in}} \]

- Saturate, wait for contrast, invert, wait for contrast, FLASH (center out)

A) \[ M_z' \text{ (just after 90°)} = 0 \quad \text{(perfect 90°)} \]

B) \[ M_z' \text{ (after TD)} = M_z^0 (1 - e^{-TD/T2}) \quad \text{(Blach term #1)} \]

C) \[ M_z' \text{ (just after invert)} = \cos \phi M_z^0 (1 - e^{-TD/T2}) \]

D) \[ M_z' \text{ (after TI)} = M_z^0 (1 - e^{-TI/T2}) + \left[ \cos \phi M_z^0 (1 - e^{-TD/T2}) \right] e^{-TI/T2} \]

\[ = M_z^0 \left[ 1 - \left( 1 - \cos \phi (1 - e^{-TD/T2}) \right) e^{-TI/T2} \right] \]

Special case TIE TD:

\[ M_z^0 \left[ 1 - e^{-TI/T2} \right] \]

\[ \rightarrow \text{using hard 180° inversion, com causal hard alpha B1 inhomogeneities (Thomas et al. '05)} \]

- after the first RF pulse:

E) \[ M_z' \text{ (just after RF)} = M_z^0 \left[ 1 - \left( 1 - \cos \phi (1 - e^{-TD/T2}) \right) e^{-TI/T2} \right] \sin \alpha \]
**MAGNETIZATION TRANSFER CONTRAST**

- Protons in macromolecules & bound to membranes have wide range of resonant freqs ("bound") \[ T_2 = 1 \text{ msec} \]

- Free protons in blood, CSF, water have narrow range of resonant freqs ("free") \[ T_2 = 50 \text{ msec} \]

- Mag transfer pulse sequence
  1) Off center freq pulse to hit "bound" (but don't hit water too hard)
  2) Wait for magnetization transfer from saturated longitudinal \( M_L \) of "bound" \[ \rightarrow M_L \text{ of "free"} \]
  3) Result of transfer \( \rightarrow \) attenuation

- N.B.: This always happens a little (cf. T1-weighted, T2-weighted)
  something to keep in mind if hard pulse (wide freq)

- Used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- View w/ MIP: maximum intensity projection along lines
  \( \rightarrow \) view as movie
**Signal-to-Noise, Contrast-to-Noise**

- Signal-to-noise defined as: \( \text{SNR} = \frac{\text{avg signal}}{\text{s.d. noise}} \)
- Temporal SNR: \( \epsilon \text{SNR} = \frac{\text{SNR}_{\text{t1}}}{\text{SNR}_{\text{t2}}} \)
- "Contrast" is a difference
- Contrast-to-noise ratio:

\[
\text{CNR}_{AB} = \frac{S_A - S_B}{\sigma_n} = \text{SNR}_A - \text{SNR}_B
\]

**Spin-Echo:**

\[
A_E = M_0 \epsilon^2 (1 - e^{-TR/T1}) e^{-TE/T2}
\]

**Gradient Echo:**

\[
A_E = \frac{M_2 \epsilon^2 (1 - e^{-TR/T1})}{1 - \cos \alpha e^{-TR/T1}}
\]

**General Rules:** Spin-Echo, long TR GE

<table>
<thead>
<tr>
<th>Protein-Density Weighted</th>
<th>TR long (no T1 diffs)</th>
<th>TE long (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR ~ T1 (big T1 diffs)</td>
<td>TE ~ T2 (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR ~ T2 (no T1 diffs)</td>
<td>TE ~ T2 (big T2 diffs)</td>
</tr>
</tbody>
</table>
Signal-to-Noise S/N

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \Delta x \Delta y \Delta z \sqrt{N_x N_y N_z \Delta t} \\
\text{voxel size} \quad \text{num repeats} \quad \text{number of voxels} \quad \text{read timestep}
\]

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  \[
  \downarrow \text{e.g., } 3\times3\times3 \text{ mm} \rightarrow 4\times4\times4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}
  \]
- More voxels (with size of voxels, \(\Delta t\) per read step constant), \(\sqrt{n}\) effect on S/N
  \[
  \downarrow \text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \sqrt{128 \times 128} = 2 \text{ times better S/N}
  \]
- # acquisitions \(\sqrt{n}\) better S/N
  \[
  \downarrow \text{e.g., } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{\sqrt{2}}{1} = 1.41 \text{ times better S/N}
  \]
- Larger timestep during readout, \(\sqrt{\Delta t}\) better S/N
  \[
  \Delta t = \frac{1}{\text{BW}_{\text{read}}} \quad \text{digitization timestep during echo acquisition}
  \]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog low-pass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \(\Delta t\)
- Must filter out freq's > \(f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
**COMPLEX ALGEBRA**

**real/imaginary**
- **add**: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
- **multiply**: \((r_1, i_1) \times (r_2, i_2) = (r_1r_2 - i_1i_2, r_1i_2 + i_1r_2)\)

**angle/phase**
- **add**: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
- **multiply**: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1A_2 \cos(\phi_1 + \phi_2), A_1A_2 \sin(\phi_1 + \phi_2))\)
- **divide**: \((A_1, \phi_1) \div (A_2, \phi_2) = (A_1A_2^{-1}, \phi_1 - \phi_2)\)

**complex to real power**: \((A, \phi)^n = (A^n, n\phi)\)

\[e^{i\phi} = \cos \phi + i \sin \phi\]

- The real "e" to "purely imaginary power"
- \(e^{i\phi^n} = (\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi\)

**Fourier transform**
- \(H(f) = \int h(t) e^{-i2\pi ft} \, dt\)
- \(H(\hat{f}) = \int h(t) e^{i2\pi ft} \, dt\)

**Convolution Theorem**
- \[\mathcal{F}[g(x) \ast h(x)] = G(k) \ast H(k)\]
  - Because of FFT, faster if kernel not small

**Conjugate**
- \(\text{N.B. : 3rd kind of vector multip. different than dot product and cross product (and G.A. non-commutative pseudoscalar multiply)}\)

**Short hand for a unit vector**
- \(\hat{e} = \text{pointing in the direction of } \phi\)

**Convolution**
- \(f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) \, dz\)
- \(\mathcal{F}[g(x) \otimes h(x)] = F(\Phi) \cdot H(\Phi)\)

- the Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each function
- How to calculate $H(f)$ for one $f$ ($f=3$):

(real signal: only need 2 correlations)

$h(t)$

real signal

imaginary signal
(zero)

$\cos(2\pi ft)$

$\sin(2\pi ft)$

$e^{-i2\pi ft}$

complex multiply

integrate/sum these multiplies across all $t$

like correlating with $\sin$ and $\cos$ (at each freq) so we get phase (at each freq.)
**Fourier transform (1b)**

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} \]
\[ = \cos (\phi) + i \sin (\phi) \]
\[ = \cos \phi - i \sin \phi \]

- \( \cos \) is an "even" function, \( \sin \) is an "odd" function

An orthogonal decomposition

- think of discretely sampled \( \sin(bx), \cos(bx) \) as vectors
- \( \text{Corr}(\vec{V}_1, \vec{V}_2) \equiv \text{projection of } \vec{V}_1 \text{ onto } \vec{V}_2 \equiv \vec{V}_1 \cdot \vec{V}_2 \)

\[
\begin{bmatrix}
\text{Corr} (\cos bx, \sin bx) = 0 \\
\text{Corr} (\sin bx, \sin bx) = 0 \\
\text{Corr} (\cos bx, \sin bx) = 0
\end{bmatrix}
\]

- in the continuous case, orthogonal functions defined as:
\[
\int_{x=hi}^{x=lo} f(x) g(x) \, dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORRELATION WITH COSINE AND SINE

- Start with a spike in the image domain.
- Take an example of a spike at \( x = 0 \) and consider \( \cos(x) \), \( \cos(2x) \), \( \cos(kx) \) all equal 1 there.
- All freqs correlate with a spike at \( x = 0 \).

- If the spike is moved away from zero, the frequency spectrum oscillates.

- To see why this is, since the spike is all zero except at spike, the dot product for a given frequency only depends on the value of \( e^{2\pi i kx} \) for \( \cos \) and \( \sin \) at location of spike.

- For positive and negative spikes (real), same distance from origin.
- For positive-negative spikes (imaginary), spikes at same distance from origin.

- One spike at distance from origin.

- This is one way of thinking about what point in k-space means, via correlating it with cosine and sine to get periodic result in image space (inverse FT).
FOURIER TRANSFORM OF AN IMAGE (2)

(1) real image
real spat. freq.

imaginary image
imag. spat. freq.

(Zero)

Fourier Transform
Inverse Fourier Transform

(2) amplitude image
ampl. spat. freq.

phase image
phase spat. freq.

(zero)

Amplitude image and phase image are equivalent representations of the complex data in the spatial frequency domain.

(3) complex vectors
complex vectors

-3 equivalent representations of image & spat. freq. space
Fourier Transform of Real Image (3)

- What a single k-space point looks like for real image (polar coordinates A, φ instead of r, θ)

**Image Space**

- Offset of stripes is k-space phase

**K-space (SpatialFreq. Space)**

- Brightness of stripes proportional to k-space amplitude
- Distance from center is stripe spacing
- Angle of point perpendicular to angle of stripes
- Value from 0 to 360°

**Inverse Fourier Transform**

(Image recon.)

**Phase**

(Should be all zero, not same as "stripe phase" above)

Cartesian dimension of k-space — x- and y- spatial freq

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin & cos — don't confuse k, kₜ w/ sin, cos!

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
FOURIER TRANSFORM OF IMAGE (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- example: cosinusoid in image space, then shifted in x-dir

REAL IMAGE

$\text{I}(x,y) = \cos(x)$

FT of $\text{I}(x,y)$

$\text{I}(x,y) = \cos(x - \frac{\pi}{4})$ → halfway between cos and sin (Shifted 45° to right)

real component less-than above because rot:

N.B.: an example of the "Fourier Shift Theorem" (see below)
FOURIER TRANSFORM OF IMAGE (S)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

\[ I(x, y) = 1 + \cos(x) \]

\[ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \]

**FT OF REAL IMAGE**

\[ H(k) = \int h(x) e^{-i2\pi k \cdot x} dx \]

\[ \text{avg image brightness} = 1 \text{ (real)} \]

\[ \text{positive center k-space} \]

\[ \text{complex} \]

[the center of k-space is zero w/ pure sin or cos image b/c avg. brightness = 0]

COMPLEX IMAGE

\[ I(x, y) = \cos(x) - i \sin(x) \]

\[ e^{-ix} \]

\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

**FT, FT^-1**

(“missing” spike results in single spike correlating with cos and sin)

N.B.: this k-space is non-Hermitian:

- k-space will only have Hermitian symmetry if image is real:
  - Hermitian symm. when complex conjugate (complex mm w/ sign flipped in imag. part) is equal to func. value w/ imag. arg.

1D:

\[ H(k) = H^*(k) \]

2D:

\[ H(-k_x, k_y) = H^*(k_x, k_y) \]

**FT OF COMPLEX IMAGE**

[Note: this is also exactly what a gradient does to image space! ]
GRADIENT COILS

- gradient coils for \( x, y, z \) generate approximately a linear gradient in the strength of the \( z \)-component of the magnetic field \( B_z \)

- for example, the \( x \) gradient coil induces a ramp in \( z \)-component of the magnetic field when moving in the \( x \)-direction:

\[
B_{G,z} = G_x x
\]

- since a pure linear gradient of \( B_{G,z} \) in only the \( x, y, z \) directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the \( x \)- and \( y \)-direction (\( B_{G,x} \) and \( B_{G,y} \))

- the other magnetic field components are usually ignored because they are so small relative to \( B_{G,z} \), since \( B_{G,z} \) is added to \( B_0 \), and since \( B_0 \) is much stronger than \( B_{G,x}, B_{G,y}, \) and \( B_{G,z} \)

- since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- the Maxwellian terms \( B_{G,x}, B_{G,y} \) are known; can be included in the re\( \text{e} \)\( \Omega \) process.
**SLICE SELECTION \((G_z)\)**

1. Slice select gradient on during RF stim

\[ B_z \]

\[ f = \frac{x}{2\pi} (B_0 + B_{G_z}) \]

2. Protons here can only be excited by a narrow band of radio frequencies.

3. To apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band).
   
   \[ = \text{sin}(\theta)/\theta \]

4. This excites protons in a narrow slab.

5. Since the slice-selection gradient introduces (space-dependent) phase shifts (see freq. encode) these have to be removed by a post-excitation rephasing \(z\)-gradient.

- Approximation from assuming tip occurs instantaneously in middle.
- Valid for small tip: \(90^\circ \rightarrow 52^\circ\).
- In practice: adjust to max, use crusher to kill spurious transverse on \(180^\circ\).
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho tapering is non-linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} p(f) e^{-i2\pi ft} df \]

(i.e., time-dependent complex (= quadrature) pulse waveform is Fourier transform of frequency spectrum of RF pulse)

Solve with: \( p(f) = \) frequency band:

\[ B_1(t) = A \cdot f_w \text{sinc}\left(\pi f_w t\right) e^{-i2\pi f_c t} \]

- amplitude controlling flip angle
- freq. width controls slice width (N.B., wider \( f_w \) is narrower sinc)
- modulation (complex) at center freq., \( f_c \)
- sinc envelope width inversely proportional to \( f_w \)

Larmor oscill at center freq.

Fourier Transform Pair, Rules

- convolution in one domain is multiplication in the other
- convolution with delta funct., impulse moves function to impulse center

Fourier Transform Solution to: \( \frac{1}{\epsilon} \)
SLICE SELECT RF PULSES

Interleaved Acquisition $\rightarrow$ better S/N b/c imperfect slice profile

Common RF pulses
- non-selective pulse ("hard" pulse)
- standard slice select sinc
- Gaussian

$\rightarrow$ pulses need to be "apodized" (have "foot" removed)
$\rightarrow$ multiply by function so begin/end of pulse is differentiable

Fat Saturation
- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
$\rightarrow$ need high quality (narrow-freq) pulse to avoid saturate water!

How to
1. Fat sat pulse
2. Wait T2 so fat signal decays, but no T1 regrowth if fat
3. RF stim for water "protons-q" interest"

Adding Another Gradient Tilts Slice

$G_x$, $G_y$, $G_z$ + $G_y$

Plane of constant $B_z$

- with 3 gradients on, can excite arbitrary angle plane
- translate plane by changing either gradient amplitude or RF freq band: $\overrightarrow{\mathbf{B}}$
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

\[
\begin{array}{ll}
\text{Spectroscopy} & \text{Imaging} \\
1) \text{ chemical shift change freq } & \rightarrow \text{ gradient changes freq,} \\
2) \text{ stimulate w/ broadband RF } & \rightarrow \text{ same} \\
3) \text{ time-sample FID containing multiple freqs } & \rightarrow \text{ same} \\
4) \text{ FT of FID to get spectrum } & \rightarrow \text{ FT of FID to get } \Delta x \text{ offsets} \\
\end{array}
\]

- this is technically correct (FT of FID) but highly misleading
- e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican turn"
- idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift-like frequencies
- rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations (which are analogous to multiple time points)
- i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>Signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>samples of oscillations in time-domain</td>
<td>FT \rightarrow \text{frequency-domain spectrum of shifts}</td>
</tr>
<tr>
<td>samples of spatial freq in freq-domain</td>
<td>FT \rightarrow \text{spatial object (like a time-domain signal)}</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because FT \approx FT^{-1} (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient \( G_x \) causes precession rates to vary linearly in x-direction

\[
\text{precession} \uparrow \quad \text{frequency} \uparrow \quad \text{in x-direction} \quad \Rightarrow \text{correct (remember that strength of } G_x \text{ causes variation of slope of } B_2 \text{ in x-direction)}
\]

- Different frequency signals are mixed together and recorded as a 1-D signal over time

\[
\Rightarrow \text{correct, but remember, we are recording summed local magnetization vectors after de-modulation}
\]

- A Fourier transform, which can convert back and forth between x-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal

\[
\Rightarrow \text{correct}
\]

- Spatial frequencies get confused/conflicated with precession frequencies

\[
\Rightarrow \text{wrong}!!
\]

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

\[
\Rightarrow \text{conceptually wrong}!!
\]

\[ F^{-1} \text{ actually converts spatial frequencies to spatial position } \]

\[ \Rightarrow \text{the spatial frequency increases for each time point in the readout} \]

\[ \Rightarrow \text{the precession freq ramp is constant each timestep} \]

N.B.: Gradient ramp does not need to be on during recording!!
FREQUENCY ENCODING (2)  
connect intuition - why phase critical

- "frequency"-encode gradient ($G_x$) turned on during
  during echo causes precession rates 
  to immediately vary with $x$-position  
  $G_x$ in x-direction  
  $G_x$ levels (= slope)  
  actually $B_z,x$  

- at beginning of gradient on, the phase of 
  signal coming from each $x$-position is the same 
  Summed phase angle is what we measure

- early after gradient on, phase advances (because 
  of faster precession frequency) arise with greatest 
  phase advance at largest $x$-position

- later during gradient on, phase advances cause 
  multiple wrap arounds of phase angle across space

- in practice, the lowest spatial frequency (= 0) 
  occurs in the middle of the gradient on time 
  because the phase is "wound" negatively by 
  a preparatory gradient (to the highest negative 
  spatial frequency) before data collection occurs 

  $\phi$  
  $\phi$ = max negative  
  $\phi$ = 0  
  $\phi$ = max positive 

  $G_x$ in x-direction  
  $G_x$ levels (= slope)  

**FREQUENCY ENCODING (3)**

why each datapoint is 1 spatial freq

**Standard Fourier transform:** (Temporal freq $\leftrightarrow$ time)

$$H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i \frac{2\pi ft}{T}} dt$$

"k" is often used instead of "f" for the frequency variable

**Imaging equation:** (Spatial freq $\leftrightarrow$ space)

$$S(f) = \int_{x = -\infty}^{x = \infty} I(x) \cdot e^{-i \frac{2\pi fx}{T}} dx$$

Sum across x of object

this is done by RF coil recording sum

Oscillations come from readout phase wrapping, where $f$ is single spatial freq (e.g., 5) and $x$ goes across object

$G_x$ [END: $G_x$; SE: $G_x (t-TE)$]

$G_x$ (t-TE)

To make image, do inverse Fourier transform of recorded signal $S(f)$

Don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each $x$ position)
ALTERNATE DERIVATION (incl. effects of $G_x$) SIGNAL EQ

- oscillators at $w = \gamma B$ at each position (just $x$ for now)

$$S(t) = M(x) e^{-i \phi(x)} \, dx$$

- by definition, freq. $w$ is rate of change of phase, $\phi$

$$\frac{d\phi(x,t)}{dt} = w(x,t) = \gamma B(x,t)$$ and integrating

$$\phi(x,t) = \int_0^t w(x,t) \, dt = \int_0^t B(x,t) \, dt$$

- assuming phase initially $0$, $B$ affected by gradients

$$B(x,t) = B_0 + G_x(t) \cdot x$$

so

$$\phi(x,t) = \gamma \int_0^t B_0 \, dt + \left[ \gamma \int_0^t G_x(t) \, dt \right] x$$

$$= \omega_0 t + 2\pi K_x(t) x$$

$k$ is time integral of gradient waveform

- demodulation removes the $B_0$-caused carrier frequency $e^{-i \omega_0 t}$ from the first equation

$$S(t) = \int_x m(x) e^{-i 2\pi K_x(t) x} \, dx$$

amplitude of each oscillator

gradient-controlled phase
PHASE-ENCODE GRADIENT $G_y$

- Turn on gradient after excitation but before readout.
- Different levels of $G_y$.
- Higher levels of $G_y$ (slope of $B_z$ in y-direction!)
- Higher spatial freq. (more phase wraps) in y-direction.
- Phase wraps persist after phase-encode gradient off.
- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase.

2D Imaging Equation

$$S(k_x, k_y) = \left[ \sum_{x} \int_{y} I(x, y) \cdot e^{-i 2\pi (k_x x + k_y y)} \right] dx \, dy$$

- Signal recorded at a single time point (one readout point).
- Complex signal (from phase-sensitive detection).
- Done by RF coil.
- Scalar (what we try to reconstruct).
- Phase angle (of scalar magnetization) in rotating frame, set by gradients.

Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia"—they stay wherever the gradients last left them.
3-D IMAGING - two phase-encode gradients

- Use $z$-gradient for 2nd phase-encoding instead of slice selection
- Excitation of whole slab (slice-select is whole brain)
- Simple spin echo example (in real life,usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

$$S(k_x, k_y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y, z) e^{-i2\pi (k_x x + k_y y + k_z z)} \, dx \, dy \, dz$$

- $I(x, y, z) = e^{-i2\pi (k_x x + k_y y + k_z z)}$

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- SNR much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

$\leftrightarrow$ phase stripes created throughout volume vs. slice:

N.B., this ignores relaxation effects for now
- Since the phase-encode gradient and the freq. encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

- Successive readout steps:
  - More rotation with higher spatial freq.
  - E.g., after y-gradient, spins at a point might be z-cycle ahead while after x-gradient spins at same point 8 cycles ahead, but counting wraps in y-direction; still only 2 ahead.

- 3D phase encode w/ G_y and G_z starts rotated in y-z plane.
Gradients Move K-space Location of Data Point

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point:

\[ k = \int_0^{t_{record}} G(t) \, dt \]

Where:
- spatial frequency recorded at t=record time
- gradient strength as function of t

Simple form:

\[ k = Gt \]

(k is area under curve)

All of the following gradients end up at the same point in k-space:

**Frequency-encode FID**

- RF 90°
- \[ G_x \]
- samples

**Frequency-encode gradient echo**

- RF 90°
- \[ G_x \]

**Frequency-encode spin-echo (plus gradient echo!!)**

- RF 90° • 180° • TE
- \[ G_x \]

**Phase-encode then frequency encode gradient echo**

- RF 90°
- \[ G_y \]
- \[ G_x \]

N.B. 180° moves to conjugate point.
\[ S(k_x, k_y) = \sqrt{\int I(x, y) e^{-i 2\pi (k_x x + k_y y)} dx \, dy} \]

\[ I(x, y) = \sqrt{\int S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} dk_x \, dk_y} \]

\[ S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} dk_x \, dk_y \]

Ideally → image is real
In practice → complex
→ use amplitude image

\[ A \rightarrow \frac{i}{\alpha} \]

Adding exponents same as multiplying two \( e^{i 2\pi k x} \)’s

Same as two sequential 1D FFTs (actual code)

\[ \sum_{k_y} \left[ \sum_{k_x} S(k_x, k_y) e^{i 2\pi k_x x} dk_x \right] e^{i 2\pi k_y y} dk_y \]

In practice, finite number of samples, \( N \) and \( M \), are collected

\[ I(x, y) = \sum_{n=-N/2}^{N/2-1} \left[ \sum_{m=-M/2}^{M/2-1} S(n, m) e^{i 2\pi n \Delta k_x x} \Delta k_x \right] e^{i 2\pi m \Delta k_y y} \Delta k_y \]
**Sampling**

- Must consider effects of sampling:
  - limited points in k-space
  - limited in range of frequencies sampled ($k_{\text{min}} \rightarrow k_{\text{max}}$)
  - limited in ratio of sampling ($\Delta k$)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

- Spatial frequency

- Correct reconstruction

- As above w/ blurring, ringing

- Aliasing occurs in spatial domain
- Replicas overlap, causing wraparound

- Thus, finer sampling of same range of spatial freqs increases FOV
**UNDER/OVER SAMPLE**

\[ \text{Fov}_x = \frac{1}{\Delta k_x} \]

\[ \delta_x = \frac{\text{Fov}_x}{N} = \frac{1}{N \Delta k_x} \]

Fov (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is Fov divided by K-space sample count

---

3 more examples (not incl. less samples to same spat. freq. Section last page)

- Basic Image
  - N=10
  - \( k_x = 5 \)
  - \( \Delta k_x = 1 \)
  - \( \text{Fov} = 1 \)
  - \( \delta_x = 0.1 \)

- Same num samp. to 2X spat. freq.
  - \( N=10 \)
  - \( k_x = 10 \)
  - \( \Delta k_x = 2 \)
  - \( \text{Fov} = 2 \)
  - \( \delta_x = 0.05 \)

- 2X num. samples to same spat. freq.
  - \( N=20 \)
  - \( k_x = 5 \)
  - \( \Delta k_x = 0.5 \)
  - \( \text{Fov} = 2 \)
  - \( \delta_x = 0.05 \)

- 2X number samples to 2X spat. freq.
  - \( N=25 \)
  - \( k_x = 10 \)
  - \( \Delta k_x = 1 \)
  - \( \text{Fov} = 1 \)
  - \( \delta_x = 0.05 \)

---

- basic image
- square pix
- x-pix half width
- replicas intrude
  - Scanner makes square image
  - "wrap" occurs
- square pix
  - twice x-pix count
  - so Fov = 2x
- this is "phase oversamp"
  - Scanner crops to square
  - replicas move out
- x-pix half width
  - twice x-pix count
  - same Fov
- this is decrease pixel size w/o change Fov
Fourier Transform Solution to Replicas

1) image/brain space
2) sampled data spatial frequency

- limit approach to Fourier transform of conv

Useful FT's
- rect
  \[ \text{Rect}(x) \xrightarrow{\mathcal{F}} W \cdot \text{sinc}(\pi x / W) \]
- Gaussian (special case)
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]
- Gaussian (adj width)
  \[ e^{-ax^2} \xrightarrow{\mathcal{F}} \frac{\sqrt{\pi /a}}{\sqrt{a}} e^{-\pi k^2 / a} \]
- comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - n / \Delta k) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p \Delta k) \]

\[ \text{Fov} = \frac{1}{\Delta k} \quad \Delta k = \frac{1}{\text{Fov}} \]
**POINT - SPREAD FUNCTION**

\[ I(x) = \Delta k \sum_{n \in \mathbb{Z}} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to $S$-function, then measured signal is:
  \[ S(m \Delta k) = 1 \]
- Substitute into $I(x)$ to get PSF:
  \[ h(x) = \Delta k \sum_{n \in \mathbb{Z}} e^{i 2\pi n \Delta k x} \]
- Simplify
  \[ h(x) = \Delta k \frac{\sin (\pi N \Delta k x)}{\sin (\pi \Delta k x)} \rightarrow \text{periodic} \]

- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in $k$-space is a sinc

**Diagram:**
- Image
- FT
- Convolve
- Multiply
- Acquisition window (truncated hi spat. b)
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int_x I(x) e^{-i2\pi k_x x} dx \]

Signal eq. \(\rightarrow\) fwd problem

\[ I(x) = \int_{k_x} S(k_x) e^{i2\pi x k_x} dk_x \]

Recon eq. \(\rightarrow\) inv. problem

\[ s = F_i \]

\[ s = \begin{bmatrix} F_i \\ k_y \end{bmatrix} \]

Linear "forward solution"

Matrix vectors have complex entries

Can build in any measurable prior

\[ F_{x,y,t} = g(x,y) e^{-i\phi(x,y)} e^{-\left((nT \pm m \Delta T + \Delta E)/T_2\right)} e^{-i\nu \Omega B(x,y) nT \pm \mu \Delta t} e^{-i(m \Delta k_x \pm n \Delta k_y)} \]

cal gain at this location

coil gain

T2 decay

\[ \Phi \] Error

(x-y dep.)

Freq. + phase

multi-coil

\[ s = \begin{bmatrix} F_{\text{coil 1}} \\ F_{\text{coil 2}} \end{bmatrix} \]

naturally incorporates undistorted field map

different sensitivity function for each coil!

contains additional info about source loc.

But, need reference scan, low-res OK

(need phase corrections for each coil?)

\[ i = F^t s \]

over-determined

More

Ponome

Inverse

\[ F^t = (F^t F)^{-1} F^t \]

(x-y)^2 \(\Rightarrow\) "small"

\[ = F^t (F F^t) \]

(x-y-coils)^2 \(\Rightarrow\) 10x bigger

for 4 coils

\[ i = [(F^t F)^{-1} F^t] s \]

slice-by-slice

assume slice select swamps others

\[ \begin{bmatrix} F^t \\ F \end{bmatrix} \begin{bmatrix} F^t \\ F \end{bmatrix}^{-1} \]
FAST SPIN ECHO (FSE)  RARE, FSE, 3D FSE

- one 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient.

- each phase “winder” is “unwound” because leftover phase would be re-focused away by 180° (vs. EPI where it persists between blips)

- the “effective TE” is the TE when center of k-space is collected (largest effect on contrast, largest echo)

- each subsequent echo has more T2 decay: \( E_n = e^{-n \text{TE}/T2} \) for \( n = 1, 2, \ldots, M \)

- by arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted

- possible to correct different T2-weighting of echoes by estimating T2 curve from \( G_y = 0 \) echo train

- 3D FSE — like 2D except wind/unwind added to thick slice select (w/commensurate 180°)
MULTI-SLAB 3DFSE, PROBLEMS

- echoes die out quickly \( \propto e^{-t/T_2} \)
- since echoes after 90° limited to \( < 30 \), can't fill 3-D k-space in a reasonable time

- SAR constraint \( \text{SAR} \propto B_0^2 \Delta f \)
  \( \Rightarrow 180° \) pulses deposit 4-6x power of 90°

- "multi-slab" is halfway between slices and single-slab

- problem at slice boundaries — esp. movement

- multislab requires slice selective RF pulses \( \Rightarrow \) longer than non-selective 'hard' pulses

- limits speed of covering k-space
SINGLE-SLAB 3D FSE


- Regular FSE (180° pulse train)

- Sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
  - This "storage" in Z-axis preserves magnetization for longer time
  - Smaller flip angles allow much longer echo trains
  - Enough to collect whole plane of 3-D k-space
  - Different than hyper echoes (not symmetric)

Contrast must consider STE

\[
SE = \sin \alpha, \sin^2 \frac{\alpha}{2} e^{-2\pi/\tau} \\
STE = \frac{1}{2} \sin \alpha \sin \alpha \sin \alpha e^{-\pi/\tau} e^{-2\pi/\tau}
\]

- Single-slab 3DFSE pulse seq.

Variable flip angle (<1 msec)

Hard (non-selective) pulse not 180°

Echo num

Echo trains

NB: time to scan k-space is \( T_{k,xy} \)

Apparent contrast time b/c of "storage"

(e.g. \( T_{k,xy} = 585 \) ms looks like FSE TE = 140 ms)
FAST GRADIENT ECHO (GRASS | FLASH | MPRAGE)
- Small tip so TR can be greatly reduced (e.g., 10 msec, less than T2)
- ‘leftover’ undecayed transverse magnetization 'unwound' and re-used "spoiled" before next shot

STEADY-STATE COHERENT (GRASS, FISP)
- Unwind phase from phase-encode Mx before next pulse (here because TR < TE)
- Unwind read gradient, too
\[ S = k \sin x \left[ \frac{1}{1 + \cos x + (1 - \cos x) T_2/T_1} \right] e^{-T_2/T_1} \]
- T2/T1-weighted contrast (bright CSF)
- Brain 0.11, fat 0.3, CSF 0.7

STEADY-STATE SPOILED (SPGR, FLASH)
- Spoil with random gradient (but this still allows some x refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

NON-STEADY STATE, MAGNETIZATION-PREP
- Preparation pulse → Strong T1-weighting
- Contrast varies in spatial-temporally-dependent way

MP-RAGE
- Longitudinal mag. not affect much by low angle pulses
- Effective TI actually time to TR that records signal
- N.B. k-space center (ky = 0)
Quantitative T1 — Intro, Methods

Motivation

- Image values are arbitrary/relative (diff seqs, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local T1 field can cause contrast variation
  - At 3T, T1 can vary by 25% across the brain sig
  - This can invert contrast in a fast gradient echo

Pre-scan normalise

- Collect low-res GE image, receive w/ body coil (no coil fall-off)
- Set params to get low GM/WM contrast
- Collect data scan (e.g. MPRAGE) w/ surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

T1 divided by T2

- MPRAGE ➞ strong T1-contrast
- SPACE ➞ T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Problems < noise in regions of low signal

MP2RAGE

1st volume ➞ PD-weighted
2nd copy of volume ➞ max T1-weighted

- N.B. SSFP-like in partition, phase-encode dir
- Convert to -0.5 to 0.5 image: \( S = \frac{\mathbf{S}^* \cdot \mathbf{S}_{T1}}{||\mathbf{S}_{T1}||^2 + ||\mathbf{S}||^2} \)
- Calc. PD & T1 from above cf. 2 flip angles
QUANTITATIVE T1 - HELMS 2-FIIP ANGLE METHOD

- start w/ gradient echo signal e.g., dropping T2-decay: \( e^{-TE/2} \)

\[
S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \frac{1 - e^{-TR/T1}}{1 - \cos \alpha \cdot e^{-TR/T1}}
\]

- Simplify / linearize to estimate

\[ TR \ll T1 \]
linear approx. of exponentials
Taylor expansion simplification of \( \sin, \cos \), drop small term

\[
S = A \cdot \alpha \cdot \frac{TR/T1}{\alpha^2/2 + TR/T1}
\]

- solve for TD and

\[ A (\text{proton-density}) \text{ given} \]
signals from 2 diff flip angles

\[ T1_{\text{est}} = 2TR \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2 \alpha_2 - S_1 \alpha_1} \]

\[ A_{\text{est}} = \frac{S_1 S_2 (\alpha_2/\alpha_1 - \alpha_1/\alpha_2)}{S_2 \alpha_2 - S_1 \alpha_1} \]

- problem: flip angle varies a lot at 3T (e.g., 25%) from nominal (requested e.g., flip series)

- acq. spin-echo and stimulated echo (EVI)

\[ S = K \cdot \sin^3 \alpha \cdot e^{-TE/T2} \]
\[ S_{\text{STE}} = \frac{1}{2} \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-TE/T2} \cdot \sin TM/T1 \]
\[ \alpha = \cos^{-1} \left( \frac{S_{\text{STE}} \cdot e^{-TM/T1}}{S_{\text{SE}}} \right) \]

Jiru & Klose (2006)

add EPI-like echo train to each FLASH excit.
**Echo Planar Imaging (EPI)**

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the G_y "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
**Spin Echo EPI**

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of oxyhemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo corrects (cancels) static $T_2^*$ ($T_2$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For TE $\approx 100$ ms, spins diffuse 10s of μm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here)

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intra/extra w/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels

N.B.: this argument applies to the extravascular signal

Over half of SE-BOLD at LST is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.

- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.

- "Asymmetric spin-echo EPI" arranges for the spin echo to occur 2 msec before the gradient echo, which gives more T2* weighting (for ky=0 echo).

- The 180° pulse rephasing reduces the T2* signal, which is why the partially rephased asymmetric spin echo has been more commonly used.

- At higher fields, spin echo EPI is more promising:
  - Signal to noise is higher so we can take spin echo hit.
  - Contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording.
**COIL FALL-OFF / UNDERSAMPLE / GRAPPA / SENSE**

- coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff fall-offs
  
  \[ \text{but what does this look like in k-space?} \]

- slow variation in RF field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space
  
  \[ \text{(N.B. not addition!)} \]

- to see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space—at all spatial frequencies!!

- simple example w/ "brain" consisting of one spatial freq:
  
  \[ \text{image domain} \]

  \[ \text{FT} \]

  \[ \text{spatial freq. domain} \]

  \[ \text{FT} \]

  \[ \text{k-space} \]

  \[ \text{FT} \]

  \[ \text{acquired image} \]

  \[ \text{FT} \]

  \[ \text{k-space} \]

  \[ \text{FT} \]

- N.B. inverse FT of k-space data "smeared" in spatial freq space is sharp image w/ fall-off (not blurred image)

- "smear" means normally orthogonal spatial freq's leak to adj. freqs.

- **GRAPPA**—construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center

- **SENSE**—general linear inverse approach

- N.B.: neither would work unless normally orthogonal spatial freqs. blurred!
- Excite multiple slices at once
- Function of $G_z$ blips is to shift slices in $G_y$ direction

This occurs because for given slice, a phase pedestal is added to the entire slice

$L \rightarrow$ This "Fourier Shift Theorem"
$L \rightarrow$ [N.B.: different than $B_0$ defect-induced incremented phase errors]

- Problem with all up $G_z$ blips $\rightarrow$ phase error builds up

**Trick #1**
- Start with 2 slices, one at $z=0$, other above
  $\Leftarrow$ If $\pi$ $(180^\circ)$ phase shift used, blip up/down same! (no effect at $z=0$)
  $\Leftarrow$ i.e., move top or bottom replica

**Trick #2**
- For multiple slices not all at $z=0$, phase no longer same for even/odd
  $\Leftarrow$ But can add phase to equilibrate to k-space before recon.

**Trick #3**
- For more than 2 slices:
  $\begin{array}{c}
  \text{1st} \quad \text{even} \quad \text{odd} \quad \text{even} \quad \text{odd} \\
  \text{1st} \quad \text{even} \quad \text{odd} \quad \text{even} \quad \text{odd}
  \end{array}$
MULTI BAND/BLIPPED C/PI (cont.)

- relation between leave-one-out aliasing and nominally fully-sampled SMS

- leave alternate lines out wraps image
- SENSE/GRAPPA to fix b/c coil view swears K-space data
- nominally, w/ SMS we record every line of K-space
- but equivalent to leave alternate out b/c our multi-slice FOV was not big enough

-- slice - GRAPPA
  - reg GRAPPA -> recon missing lines
  - slice GRAPPA -> recon multiple k-spaces i.e. not

  For each overlapped slices by training on fully-sampled data at beginning of scan

- inter-slice "leakage block"

  - when training GRAPPA kernel on fully-sampled data, also minimize inter-slice leakage (split-slice-GRAPPA)

  - can also do regular GRAPPA on top of this
  - reason: for diffusion, loss in S/N from undersample cancelled by shorter TE readout

  - gain from reduced image distortion from shorter readout
ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- entire k-space must be filled before 3D image is reconstructed
- since entire volume is excited each shot, potentially higher S/N
- must use smaller flip angle to avoid killing $M_L$ since entire volume excited every partition (e.g., every 80 ms/dec)

- main issue is movement artifact since data assembled from many shots over several secs
- breathing-induced B0 problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less
  gradient power required than w/ trapezoids (less eddy currents)

earlier EPI hardware like this : sinusoidal gradient waveform
from resonant circuit w/ non-uniform sampling to get constant Δk

- sinusoids in both Gx and Gy give spiral k-space trajectory

- constant angular velocity goes too fast at large kx, ky
- constant linear velocity better but impractical near kx=0, ky=0
- compromise : start constant angular, end constant linear

\[
\begin{align*}
\text{Constant angular velocity} \\
& w(t) = \omega_0 t \\
& k(t) = A t e^{i \omega_0 t} \\
& G(t) = \frac{1}{\tau} \frac{d}{dt} k(t) \\
& = A e^{i \omega_0 t} + A \omega_0 e^{i \omega_0 t} \\
& G_x(t) = A \cos \omega_0 t - A t \omega_0 \sin \omega_0 t \\
& G_y(t) = A \sin \omega_0 t + A t \omega_0 \cos \omega_0 t
\end{align*}
\]

\[
\begin{align*}
\text{Constant linear velocity} \\
& w(t) = \omega_0 T_e \\
& k(t) = A T_e e^{i \omega_0 T_e} \\
& G(t) = \frac{1}{\tau} \frac{d}{dt} k(t) \\
& = A e^{i \omega_0 T_e} + \frac{A}{2} \omega_0 e^{i \omega_0 T_e} \\
& G_x(t) = \frac{A}{\tau} \cos \omega_0 T_e + \frac{A}{2} \omega_0 \cos \omega_0 T_e \\
& G_y(t) = \frac{A}{\tau} \sin \omega_0 T_e + \frac{A}{2} \omega_0 \sin \omega_0 T_e
\end{align*}
\]
**Spiral 3D IR FSE**

(from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)

- Possible to present sign
- High, uniform contrast, but lots of waiting (TI), high BW

RF

180° (prep1) → TI = 700 msec

180°

Crush + wind

180°

Crush + unwind

(roll select)

FID

Echo 1

Echo 2

k-space

("stack of spirals")

3D

Loop order

Spiral interleaves

k₂ interleaves

k₂ echoes

Echoes → (after one 90°)
**PHASE ERRORS & ECHO-CENTERING ERRORS**

Anything that causes a deviation of the $B_z$ field strength from the expected value ($B_{o,z} + G_{x,z} x + G_{y,z} y + G_{z,z} z$) changes precession frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

**Fourier shift theorem**

Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x-x_0) = \int \frac{e^{-i2\pi k_x x}}{k_x} S(k_x) e^{i2\pi k_x x} dk_x$$

- Correct with shimming and $B_0$-mapping/phase unwrapping before reconstruction.

**Echo Centering Error**

- If realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.
- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction. Magnitude image unchanged.

- Fourier freq. shift theorem: freq. shift in freq. domain causes phase shift in spatial.

- Offset in spatial freq. space.
FAST SCAN ARTIFACTS  EPI vs. Spiral

brain-induced field defects lead to phase errors

EPI

- $G_x$ readout gradient strong → field defects smaller percentage
  less deformation of $k_x$ (vertical stripe components)

- $G_y$ "blips" are weak and total $G_y$ readout time
  much longer (5 times) than standard readout (50 ms vs. 10 ms)

- an extra gradient in the $x$-direction for example, maps
  and unmaps phase as a function of $x$-position

- but $G_x$ big, so effect in freq.-encode direction is much
  less than in phase-encode direction, where error accumulates

- for a given $x$-position, the strength of the spurious gradient
  is constant, so the accumulation of phase error results
  in a shift in the $y$-direction ($k_x$-space spin-stripe disp)
  (N.B. Shift varies w/x-position)

Spiral

- with center-out spirals phase errors accumulate
  in a radial direction

- thus, higher spatial frequencies have more error (= more shearing)

- for spurious $x$-direction gradient as above, there is
  a radial blurring, rather than a vertical shift
  because higher frequency phase stripes misaligned
  relative to low spatial freq

- for defects with more complex contents in the $y$-direction
  (than linear, as above) the vertical shifts (in EPI) will
  vary with $y$-position, and may result in signals from different
  $y$-positions being reconstructed on top of each other
IMAGE-SPACE VIEW OF LOCALIZED $\phi$ DEFECT, EFFECT ON RECON

- Localized $\phi$ defects often arise from air pockets embedded in tissue
  - Air in middle/outer ear $\rightarrow$ indentation in inferior temporal lobe
  - Air under olfactory epithelium $\rightarrow$ orbitofrontal cvx, ant, thal. compression

- Collect one data (k-space) point
  - 4 cycles of phase in y-dir ($\phi$-gradient)
  - Localized $\Delta\phi$ defect
  - Complex multiply $= \text{correlate sin, cos with brain}$
  - Brain structure sampled with distorted stripes
  - $\text{one complex number}$

- Reconstruction from distorted data points
  - ... + $\begin{bmatrix} \text{undistorted} \\ \text{stripes used by inverse FFT} \end{bmatrix} \times \text{amplitude and phase of this component}$
  - $\begin{bmatrix} \text{same for } 5 \\ \text{cycles} \end{bmatrix} + \ldots = \text{brain}$

- Local upward displacement image phase (phase encode dir)

**N.B.:** image shift only occurs if shift spiky traces sampled w/ successively later echos times (see next page)

- Same defect makes leftward dent in vertical phase stripes

- Spatial information can be lost when continuous changes in phase are flattened by $\phi$ defect

- Shifts can pile multiple pixels on top of each other into one bright pixel

- Local estimates of $\Delta\phi$ needed to correct images
  1) Fieldmap method: shift each image pixel proportional to $\Delta\phi$ in phase-encode dir.
  2) Point-spread-function: extra phase encode to estimate PSF (should be $\delta$-function)
     deconvolve distorted image in phase-encode direction
LOCALIZED B₀ DEFECT, EFFECT ON RECON

- When local B₀ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spat. freq.)

- If each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space

- A k-space freq. offset only results in image space phase shift (Fourier freq., shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- However, with w/EPI, static B₀ defect causes more and more local displacement of image phase stripes for each additional ky line

  - That is, later lines have greater spat. freq. offset
  - Effectively stretches k-space in ky direction
  - Same num samples to higher spatial freq. shrinks FOV (squishes voxels - see FOV page)

- When image is reconstructed, region with local B₀ defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:
  1) Static local ΔB₀ defect
  2) Successive increases in phase error for successive spat. freq. measurements during long EPI readout
  3) Small size of ky phase encode blips relative to B₀ defect (much less of this effect in freq. encode direction)

- Respiration (which affect B₀) in 3D FLASH might cause similar effect within k₂ partition (if successive spat. freqs.)
GRADIENT NON-LINEARITIES

- ideally the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto the z-component of the B Field — $B_z$ — in the x, y, and z-directions.
- in practice, gradient coils are non-linear (esp. printed-circuit-like).
- non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- non-linearities result in phase errors, which result in 3-D image distortion:
  - a non-linear slice-select gradient will excite a curved slice
  - non-linear phase and frequency encode gradients will distort in-plane features

- some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in plane), and EPI scans (no corrections!).
- this can result in errors approaching 1 cm in function-structure overlays.
- different coil designs have different directions of distortion (!)

- the assumption of non-Maxwellian gradients results in additional phase errors.
- these can also be corrected since the $B_x$ and $B_y$ components are known.

These effects do not build up over time in phase-encode directions since they only occur when gradients are turned on.

Fourier shift theorem

These distortions are predictable and can be corrected.

That is, the assumption that gradients cause no field
in the $B_x$ and $B_y$ direction.
SHIMMING AND $B_0$-MAPPING

- Passive iron shims inserted to flatten $B_0$ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $B_0$ field (a few ppm good)

- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc.) (= several hundred ppm)

- Linear shim coils impose gradients in $x$, $y$, and $z$
- Higher order shims impose higher order spherical harmonic field components (e.g. $z^2$)

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $B_0$ field

- Local resonance offsets caused by $B_0$ defects estimated from images
  - e.g., sample phase at multiple echo times

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  - This only corrects spatially gradual field defects
  - Local defects due to air in sinuses much higher order than shims

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to un-warp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq-encode gradient is strong relative to defects)
**NAVIGATOR ECHOBES**

- **1D navigator**
  - Bφ drift problem: slow up/down drifts in Bφ continuously occur.
  - A pedestal in Bφ is not a phase (Fourier shift theorem).
  - In EPI, mainly affects phase-encode dir b/c small drift in readout.
  - Result is successive volumes drift in phase-encode dir.

**Gradient balance problem**

- Unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space, causing N/2 (Nyquist) ghosting. 
  - Another phase error.

- **3D navigator**: Collect 3D sphere in k-space.
  - Rotating object → rotation of k-space amplitude pattern.
  - Translation of object → phase shift of k-space phase (Fourier shift).
  - Sample at sufficient radius to pick up high spatial freq features.
  - N.B.: Excite whole volume.
  - Do NS hemispheres separately (less T2*, cancel EPI-like error accumulation).

Walth et al. (2002) MRM

\[ x(n) = \sin\left(\frac{\pi}{T} \sin^{-1}\left(\frac{z(n)}{T}\right)\right) \]
\[ y(n) = \cos\left(\frac{\pi}{T} \sin^{-1}\left(\frac{z(n)}{T}\right)\right) \]
\[ x(n) = \frac{2n - N - 1}{N} \]

RF

- Can be used for prospective motion correction (rotate, translate with gradients).
- Better estimate because of speed than full TR & EPI images (27 ms vs. 2.4 sec).
- May need to smooth rot, trans estimates across time (e.g. Kalman filter).
RF FIELD INHOMogeneITIES $B_0$ inhomoegenities

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way
  - variations can be used (cf. GRAPPA, SENSE) and/or corrected

- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  - potentially worse (why local transmit is still in progress)
  - usu. fixed by using a large transmit coil (e.g. body coil)

- RF penetration at higher fields ($\leq$ higher RF frequencies)
  is less uniform:
  1) decreased RF wavelength (closer to size of head) at higher freq.
  2) increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

  normalization ("pre-scan normalize")
  - record low-res volume (b/c coil fall-off is smooth)
    through both body coil and small coil(s)
  - divide small coil/body coil at each voxel to determine receive field
  - use receive field to normalize main image(s)
  [see also: $T_1$, MP2RAGE, $T_1/T_2$]
**Diffusion – Weighted Imaging**

Simple diffusion weighting

- Apparent diffusion coefficient map
- To get large b, need G^1, G^2^ (need big G’s)
- Long G^1 gives spurious T2-weighting
- Can use stimulated echoes: 90° RF → G^1 → 90° RF → G^2 → Transverse
- Diffusion image from b = 0, b = large (adder’s better)

1) Anisotropic Diffusion (Gaussian)

- Measure D along multiple axes
- Have to measure tensor, not scalar
- Even for determining one primary direction

\[ D = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \]

D := u^T ∙ D ∙ u

Scalar diffusion - diffusion tensor - measurement direction

**Diffusion Surface (Non-Gaussian)**

- Need to measure diffusion in many directions (≥6) to properly characterize even 2 main directions

2) Length scale by multiple b-values

- Fit line to semi-log signal as function of b
- If not straight line: multi-exponential, e.g., bi-exponential curve: 
  \[ S = A_1 e^{-bD_1} + A_2 e^{-bD_2} \]

**Tract Tracing**

1) Markov process
2) Crossing fibers
3) “Freeway ramp” prob
4) Sharp turns (into gyri)

**Diffusion Tensor Imaging**

- Classical diffusion coefficient
- Spins acquire phase during first T
- If spins diffuse (move) along gradient by time T, signal is lost because negative G^1 doesn’t re-phase
- Attenuation: \[ A(D) = \frac{S_0}{S} = e^{-bD} \]
  \[ b = \gamma^2 G^2 G^1 \]
  \[ (T-\frac{\Delta}{2}) \]

**Log-Signal vs. Diffusion Tensor**

- Voxels
- Tract Tracing

(Mulkern, 1999)

(e.g. assume A_1 = A_2 = 0.5)

Data both b-vec b-value log sig curve
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)
  - allows longer TE
  - flips M₂ so rephase gradient same sign as dephase

- eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/particle time constants)
  - (also, keep Fourier orthogonal to diffusion-encoding gradients)

- Nagy et al., (2014) MRM

- Twice-refocused Spin-echo (for center k-space)

RFₘᵟ ← 90° ← RFₘᵟ

G₁₂

Gᵧ

Gₓ

RFₘᵟ

TE eff

[expanded in time for clarity]

TE eff/2

Phase dispersion (6 echo)

Y₁₆SE = 0 = Y₁ - Y₂ - Y₃ + Y₄

RF₁₂, ignored, used as 'fid' for STE TRSE

SE₁,₂

SE₂,₃

SE₁,₃
PERFUSION - ARTERIAL SPIN LABEL

- Basic idea:
  - Tag blood below area of interest
  - Collect control & tagged image
  - Assume directional input flow

  Continuous ASL (CASL) - [continuously tag a plane]
  Pseudo-continuous ASL (pCASL) - [greatest on, blood gets adiabatically inverted as it passes through location w/corresponding resonance tag]

  Pulsed ASL (PASL) - e.g., EPSTAR, FAIR, PICORE, QUIPPS II
    - Tag block of tissue below slice(s)

- Small diff between control and tag (~1%)
  - Requires accurate balancing of control & tag images, control mag. transfer

- Contrast problems:
  - Transit delays - biggest confounding factor
  - Relaxation rate diff.
  - Venous clearance (vs. microphages, which get stuck!)

- QUIPPS II - Quantitative perfusion
  1) Pre-saturate spins in target slices
  2) Tag - 180° pulse below slices
    - Control - 180° pulse above slices (to control off-resonance)
  3) Saturate tagged block to end tag (T1)
    - Can use train of thin slices pulses at top of tag band
  4) EPI or spiral images of target slices (T2)
    - Image most distal slice last to cancel delays
    - Fast between slice so imaging excitations don’t get interpreted as flow

\[ \Delta M = \text{flow} \times \left[ 2M_0 \text{~} T1, \text{~} e^{-T2/T1} \right] \]

- Can extract flow and BOLD
  - Adjacent subregions minimize movement artifact

1) Alternate tag and control, GRE TE = 30 ms
  - Control + tag = BOLD
  - Tag - control - tag - control...

2) Dual echo spiral
  - k = 0 early => hi S/N flow
  - TE = 30 ms => BOLD
PERFUSION - pCASL

- Original CASL (continuous arterial spin labeling) requires
  RF on continuously to adiabatically invert blooded flowing
  through one plane
  
  \[ \Rightarrow \text{can only image one slice (b/c dephasing from gradient)} \]
  
  \[ \Rightarrow \text{hard to keep RF on continuously on modern scanner (esp. BC)} \]
  
  \[ \Rightarrow \text{can use special purpose RF transmit (separate xmit channel)} \]

A) Original CASL

RF

[Image formation module (readout) \[\rightarrow \text{multiple possibilities}\]

Gz

(on pulsed!)

B) pCASL - pseudo continuous arterial spin labeling Dai, Alsop (2008)

RF

Gz

begin readout

\[ \text{comb}(t) = \sum_{n} \delta(t-n) \]

\[ \text{rect}(t) = \begin{cases} 1 \text{ if } |t| < \frac{1}{4} \\ 0 \text{ otherwise} \end{cases} \]

\[ \text{For constant } G_z: z = \frac{n}{G_z \Delta t} \]

\[ \Rightarrow \text{aliased labeling planes at: } \delta = n/\Delta t \text{ in frequency space, modulated by \text{sinc}(\pi \delta t)} \]

\[ \text{Use Hamming or hyperbolic secant to reduce replicas} \]

\[ \text{Hamming} \rightarrow \text{FT} \rightarrow \text{Hyperbolic secant (cf. Gaussian)} \]

C) pCASL w/ shaped gradients

RF

Gz

0.8 msec tag

FLASH

Readout

EPI

SMS

Stack spirals 3D

- Tag pulses have phase offset respecting gradient
- Control identical except every other has pi phase
- no net flip
OFF RESONANCE EXCITATION

- Main idea: examine evolution of $\vec{M}$ vector in rotating coord syst set to "off-resonance" $\vec{B}_1$ field freq (wrf), not Larmor freq of $\vec{M}$ ($\omega_0$)

- Normally, if rotating coord syst freq set to Larmor freq ($\omega_{\text{rf}} = \omega_0$), an actually precessing $\vec{M}$ will be stationary (ignoring decay) $\Rightarrow$ implies effective $B_z = 0$ in rotating

- Now, move $\vec{M}$ to rotating coord syst at $\vec{B}_1$ freq lower than $\omega_0$ (assume $\vec{B}_1 = 0 = \phi$): existing $\vec{M}$ will now appear to precess around z-axis:

  N.B.: this is precession in already rotating coordinate system!
  $\omega_0 = \omega_0 - \omega_{\text{rf}}$

  freq of precession in rotating coordinate syst
  Larmor rotation freq of $\vec{M}$
  $\Rightarrow$ incorrectly set rotating coord syst freq

- Thus, viewing $\vec{M}$ vector in off-resonance rotating coord syst makes it look like additional $\vec{B}_2$ field is causing "extra" precession

- "Extra" $\vec{B}_2$ component is proportional to $\Delta \omega_0$ offset $\Rightarrow$ can be pos or neg: too low $\Rightarrow$ pos $\vec{B}_2$

- Extra $\vec{B}_2$ adds to $\vec{B}_1$ resulting in slow precession around tipped axis: $\vec{B}_{\text{eff}}$ (effective)

- Extra z-gradient can have same effect on $\Delta \omega_0$ (changes $\omega_0$ instead of changing $\omega_{\text{rf}}$)

  $\vec{B}_{\text{eff}} = \left( \frac{\Delta \omega_0}{\gamma} \right) \hat{k} + B_\phi \hat{k} + B_1 \hat{i}$

- Adiabatic RF pulse

- RF: sweep freq
- $\omega_0$: constant
- RF: const freq
- $\omega_0$: sweeps because spins flow along gradient direction
SPECTROSCOPY + IMAGE

- **Chemical shift**: small displacement resonant freq due to shielding of target nucleus (e.g. \( ^1H \)) by surrounding electron orbitals.

  - e.g., acetic acid: oxygen attracts electron so less shielding of target nucleus

- How we get Chemical shift spectrum:
  
  - Data before FT is a series of time-domain samples of the mix of shifted-freq offsets
  
  - FT turns data into "shift spectrum"

Larmor oscillations are multiplied (PSD) by center freq to obtain \( \Delta f \) (not MHz high freq)

- N.B.: opposite "direction" of FTs!

### Table: NMR vs MRI

<table>
<thead>
<tr>
<th>Signal</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMR</td>
<td>Time domain FT freq spectrum shift</td>
</tr>
<tr>
<td>MRI</td>
<td>Spatial FT spatial object freq samples (like time-domain signal)</td>
</tr>
</tbody>
</table>

Pulse Sequence

- Since we are already using phase (freq) encoding for space, we need an "extra dimension" w/ all gradients OFF!

- Use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal \( \rightarrow \) and FT-it like chemists do!
PRESS, MEGA-PRESS

 usu. single voxel by using 3 orthog. slice select
(tho can add PG gradients & more excitations to get multiple vox)

PRESS — 3 orthog. slice select

MEGA-PRESS — add "editing" RFs to suppress solvent (water)

MEGA 180° pulses set to freq. of solvent

Crushers — asymmetric spoilers to dephase spins in bandwidth of selective MEGA pulses

$G_1$, $G_3$ — symmetric spoilers to dephase spins in bandwidth of selective MEGA pulses

only record 2nd half

FT to get shift spectrum

FT to get shift spectrum
PHASE-ENCODED STIMULUS & ANALYSIS

Periodic stimuli (phase-encoded) - e.g., 8 cycles at 64 sec/cycle

Calculate significance
- ratio between amplitude at stimulus frequency (=signal)
  and average of amplitudes at other frequencies (=noise)
- ignore harmonics, low freq (=movement)

Smooth
- vector average of complex significance (A, \( \phi \)) with that at nearest neighbor surface points

Display
- plot phase using hue and saturation to indicate significance

Delay correction
- record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
- vector average after reversing angle of one
  penalizes inconsistent more than just any of angles

Typically 0.5-5% amplitude

Strongly periodically activated single voxel time course

Remove constant (avg) and linear trend

FFT, convert to A, \( \phi \)

Reversed CCW

CCW significance (complex)
CONVOLUTION

\[ h(x) = f(x) \ast g(x) = \int_{-\infty}^{+\infty} f(z) \cdot g(x-z) \, dz \]

- definition of convolution \((f \ast g)(x)\)
- commutative

\[ f(z) \cdot g(x-z) \]

\[ g(x-z) \]

**Why reverse makes sense**

blc commutative, this is thinking like: \( \int g(x) \cdot f(x-z) \, dz \)

impulse response function (HDR)

impulses (expt. design)

**Intuitive non-reversed view of convolution output**

\( g(x+z) \) instead of \( g(x-z) \)

NB. Cross-corr
Same as convolution except no reversal

NB. Auto-corr
Same, except no reversal and use same fun for both \( f, g \)

start here!

How to calculate convolution output for this time point (only 3 terms in sum, all other zero)
**GENERAL LINEAR MODEL**

\[ \hat{y} = \hat{X}\hat{h} + \hat{s}\hat{b} + \hat{n} \]

- **data** = design, HDR + drifts, weights + noise

- **goal** is to solve for the hemodynamic response functions, \( \hat{h} \)

- **first**: preconvolve
  - convolve \( \hat{X} \) with \( \hat{h} \)
  - solve for \( \hat{\beta} \)
  - \( \hat{y} = \hat{X}\hat{\beta} + \hat{n} \)

- **simpler**: preconvolve
  - convolve \( \hat{X} \) with \( \hat{h} \)
  - solve for \( \hat{\beta} \)
  - \( \hat{y} = \hat{X}\hat{\beta} + \hat{n} \)

- **multiple conditions**
  - \( \text{cond}_1 \) occurs
  - \( \text{cond}_2 \) occurs
  - \( \text{cond}_1 \) re-occurs

- **maximum likelihood estimate** (Liu et al., 2001 *Neuroimage*)

1. Assume white noise, solve for \( \hat{h} \)

2. \( \hat{h} = (X^TP_sX)^{-1}X^T(P_s\hat{y}) \) where \( P_s = I - S(s^TS)^{-1}s^T \)

   \( \Rightarrow \) projection matrix that removes part of vector that lies in \( S \) space

3. Significance (how to construct F-ratio)

\[ F = \frac{N - K - L}{K} \left[ \frac{y^T(P_{xs}-P_s)y}{y^T(I-P_{xs})y} \right] \]

- \( P_{xs} \) - projection data on \( \hat{X} \) + nuisance subspace
- \( P_s \) - projection data on nuisance subspace
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- With no nuisance functions ($S$), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

$$\hat{y} = \hat{X}\hat{h} + \hat{e}$$  data exist for error

$$\hat{X}\hat{h} = P_x\hat{y}$$  projection matrix, $P_x$, operates on $y$ to give projection of data into experiment space, $X$

- When nuisance functions, $S$, are considered, problem: $S$ may not be orthogonal to $X$

For example: linear trend not orthogonal to std. block design.

Remember: "orthogonal" means $\text{dot prod} = 0$ [cor = 0]

Geom $X$: space of data modeled by all reference and nuisance.

Orthogonal projection onto nuisance ($P_S y$) (data explained by $S$)

Oblique projection onto reference ($P_X y$) (data explained by reference)

Orthogonal projection onto reference and nuisance ($P_{X-S} y$) (same as projection onto reference only in special case where $S \perp X$)
SEGMENTATION & SURFACE RECON

1) MNI auto-Talairach → generates 4x4 matrix
- make average brain target (blurry)
- blur target (further), blur single brain (a lot), gradient descent on xcorr
- repeat w/ less blurring of avg target and current brain
- problems: variable neck cut-off
  → but much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")
- histogram of pixel values in 10 mm thick HOR slices
- smooth histogram
- peak find to get initial estimate of white matter
- discard outlier peaks across slices
- fit splines to peaks across slices
  → interpreted scaling factor 1 to HOR
- scale each pixel so WM peak is 110
- refine estimate to interpolate in 3D
  → find points in 5x5x5 within 10% of WM, get new scale for them
  build Voronoi to interpolate scales upset above
  soap-bubbles-smooth Voronoi boundaries (3 iterations)
  re-scale each voxel

3) Skull Stripping (output: "brain")
- "shrink-wrap" algorithm
- start with ellipsoidal template
- minimize brain penetration and curvature
  - curvature: spring force
    (from center-to-neighbor vector sum)
  - brain penetration
    apply force along surface normal that prevents surface from entering gray matter

- decompose into L and tangential (local normal from summed normal cross products)
- Implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")
- More formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[ \mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t) \]

For one vertex

\[ \mathbf{F}_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (I - n_{\text{center}} n_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \]

Identity +3

\[ \text{stronger than normal (0.5)} \]

Identity *3

\[ \text{vector to neighbor vertex} \]

\[ \text{project second onto normal} \]

\[ \text{first} \]

\[ \text{normal} \rightarrow \text{tangential} \]

\[ \lambda_{\text{normal}} \left[ \sum_{\text{neigh}} (n_{\text{center}} n_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) - \frac{1}{\text{#vertices}} \sum_{\text{neigh}} \sum_{\text{neigh}} (n_i n_j) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_v) \right] \]

\[ \text{average normal component} \]

\[ \text{projection of a neighbor or vertex} \]

\[ \text{vector onto normal in the direction of the normal (n is squared (as shown) so we get a vector out (not a scalar))} \]

\[ \text{vector subtract off} \]

\[ \mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \frac{\mathbf{n}_{\text{center}}}{d} \max_{\text{d}} \left[ 0, \tanh \left[ I \left( \mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}} \right) - I_{\text{thresh}} \right] \right] \]

Intensity MRI data

\[ \text{d sample points into brain along the direction of normal} \]

\[ \text{max force is saturated at 1.0} \]

\[ \text{max force product = 1.0} \]

\[ \text{outside (dark)} \]

\[ \text{skin (light)} \]

\[ \text{skull (dark-light-dark)} \]

\[ \text{Surface (moving outward)} \]

\[ \text{GM} \]

\[ \text{WM} \]

\[ \text{Ideal skull strip} \]

Snapshot of surface and "core sample" from one vertex
SEGMENTATION & SURFACE RECON

4) Non-isotropic filtering (output: "win") — "floss" and "speckle"
- preliminary hard thresholds: output
- find ambiguous/boundary voxels
  \( \Rightarrow \) 20\% or more of 26 immediate neighbors different
- find plane of least variance
  for each ambiguous voxel
  \[ \text{consider } 5 \times 5 \times 5 \text{ volume around 1 voxel} \]
  \[ \text{find plane of least variance in this hemisphere} \]
  \[ \text{median filter w/hysteresis} \]
  \[ \Rightarrow \text{if } 60\% \text{ of within-slab differ, reverse classification} \]
  \[ \Rightarrow \text{"flosses" sulci without blurring} \]

5) Find cutting planes
- callosum, to separate hemispheres (SAG)
- midbrain, to avoid fill into cerebellum (T1R)
  \( \Rightarrow \) Talairach to start:
  fill WM in SAG or T1R till min area

6) Region-growing to define connected parts (output: "filled")
- inside-out, outside-in, inside-out — for each hemisphere
  \( \Rightarrow \) "wormhole filter." \( (3 \times 3 \times 3 = \text{center} + 26) \)
  \[ \Rightarrow \text{fill (unfilled) voxels if } 66\% \text{ neighbors differ} \]
7) Surface Tessellation (output: rh.orig, lh.orig)

- Find filled voxels bordering unfilled
- Make ordered list of neighboring vertices
  (to cross-products oriented properly)
- Long list of values associated with each numbered vertex
  e.g. position (orig, morphed), area (orig, morphed),
    curvature (intrinsic, Gaussian), "sulcuesness"
    (summed 1 movement during unfolding),
    cortical thickness,
    fMRI data
    EEG/MEG dipole strength

- Separate fMRI data set must be aligned, sampled
  fMRI voxels larger
  Sample at each surface vertex
  nearest-neighbor "soap bubble" smoothing
  to interpolate data into hi-res mesh

- Some quantities only well-defined on surface
  gradient of magnitude of
cortical map measure (e.g., eccentricity)
SEGMENTATION & SURFACE RECON
Smooth, inflate, final surfaces

-smoothing/inflation/WM, pial done as derivative of energy functional

\[
\mathcal{J} = \mathcal{J}_{\text{tangential}} + \lambda_{\text{normal}} \mathcal{J}_{\text{normal}} + \lambda_{\text{image}} \mathcal{J}_{\text{image}}
\]

- total scalar error to minimize
- scalar tangential error (fixed by redistributing vertices)
- scalar normal error (fixed by reducing curvature)
- scalar image error (fixed by moving toward target image value)

\[
\mathcal{J}_{\text{normal}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2
\]

\[\text{across all vertices} \]

\[\frac{1}{2} \text{ so no coefficient on derivative}\]

\[\text{across all vertices of one vertex}\]

\[\text{vector unit normal} \]

\[\text{vector from current center to one neighbor (position vector diff)} \]

\[\mathbf{t}^x_{\text{center}}, \mathbf{t}^y_{\text{center}} \]

\[\text{first 2 eigenvec. of neighbor vector cloud (\(n\) is third)} \]

\[\mathcal{J}_{\text{tangential}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2 + \left[ \mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2 \]

\[\text{"squishing" or "wash"} \]

\[\text{vector in tangent plane} \]

\[\text{project vector to neighbor onto \(x\) \& \(y\)} \]

\[\text{direction in tangent plane} \]

\[\text{I}_{\text{targ}} \text{ for WM: mean of voreses labeled WM in 5 mm neighborhood} \]

\[\text{I}_{\text{targ}} \text{ for pia: global - small num for CSF - like} \]

- take directional derivative of energy functional (to find steepest uphill)

- move each vertex in the opposite (negative) direction w/ self-interest test

\[- \frac{\partial \mathcal{J}}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{targ}} - I(\mathbf{r}_{\text{center}}) \right] \nabla I(\mathbf{r}_{\text{center}}) \]

\[\nabla \text{ goes b/c const} \]

\[\sum_{\text{neighbors}} \lambda_{\text{normal}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right] \mathbf{n}_{\text{center}} \]

\[\text{x-component of tangential} \]

\[\text{vector - calculate gradient on image (first blur w/Gaussian)} \]

\[\text{scaled by unit normal vector} \]

\[\sum_{\text{neighbors}} \left[ \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right] \mathbf{t}^x_{\text{center}} + \left[ \mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right] \mathbf{t}^y_{\text{center}} \]

\[\nabla \text{ by \(C_1 \) \& \(C_2\)} \]

N.B.: eq. 9 in Lella, Fisch & Siden - different - and incorrect!

HOW TO derive:

\[\frac{\partial \mathcal{J}}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{targ}} - I(\mathbf{r}_{\text{center}}) \right] \nabla I(\mathbf{r}_{\text{center}}) \]

\[\text{constant \& var} \]

\[\lambda (\mathbf{u} \cdot (-\nabla C)) \]

\[\lambda (C - C_{\text{targ}}) \]

\[\nabla C_{\text{targ}} = \nabla C \]

\[\nabla C \cdot \mathbf{n} = 0 \text{ (normal)} \]

\[\text{and other relations like} \]

\[\text{for \(C_1 \& C_2\)} \]
SULCUS-BASED CROSS-SUB. ALIGN

- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"
- Add term to energy function: "sulcus-ness" error: \((S_{\text{curt}} - S_{\text{targ}})^2\)
- Bootstrap by morph to one brain, make any target, remorph to avg targ

![Diagram showing the process of sulcus-based cross-subject alignment.](image)

- Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)
- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal crinkles (for average "inflated")
  - Inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")
- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

\(\Rightarrow\) N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. sub.)
SOURCE OF EEG/MEG

PSPs
- anisotropic cables
  - aligned spatially
  - coherent/biased stim on one end
- isotropic
  - "closed field"
  - (invisible at distance)
- no distant signal from axon spike
  - output too close

N.B.: spikes only detected by 15μm microelectrode in gray matter!

Head
1) Local dipole
2) EEG through skull, skin
3) Sweating because skull ½th conductivity of brain

MEG
- Radial dipoles lost
- Tangential dipole generates Gabor-like scalp distrib. of B field
INTRACORTICAL CIRCUITS & ORIGIN OF EEG

Cell types
- excitatory (spiny)
  - pyramidal
  - spiny stellate (e.g., V1 layer 4c)
- inhibitory (smooth)
  - basket
  - double bouquet
  - chandelier
  - clutch

Circuits
- huge complexity
- first principal components: input \rightarrow layer 4 \rightarrow layer 2/3 \rightarrow feedforward
- microelectrode recording (e.g., 10 \mu m tip):
  - high pass \rightarrow spikes
  - low pass \rightarrow local field potentials
- spikes only recordable in gray matter
- white matter spikes only recordable with pipette w/ very fine tip because inward & outward currents so spatially close in axon/spike (> 1 \mu m)

intra/inter cortical connections cartoon

"lower" (e.g., V1)
- 2/3 feedforward
- 4 input
- 5 motor output
- 6 feedback

"higher" (e.g., V2)
- 2/3
- 4
- 5
- 6

ascending input (e.g., dLGN)
output to sup. collic.
feedback avoids layer 4
motor striatum
to higher areas
**GRADIENT, DIVERGENCE, CURL**

**Gradient** \( \mathbf{\nabla} \) (generalized derivative)

\[
\mathbf{\nabla} s(\mathbf{r}) = \frac{\partial s(\mathbf{r})}{\partial x} \mathbf{i} + \frac{\partial s(\mathbf{r})}{\partial y} \mathbf{j} + \frac{\partial s(\mathbf{r})}{\partial z} \mathbf{k}
\]

- Turns scalar field into vector field
- Scalar function defined at each \( x, y, z \) point \( \mathbf{r} \)
- Change of \( s \) in \( x \) direction at point \( \mathbf{r} \)
- Unit vector in \( x \)-dir

**Divergence** \( \mathbf{\nabla} \cdot \mathbf{v}(\mathbf{r}) \) (derivative "dot product")

\[
\mathbf{\nabla} \cdot \mathbf{v}(\mathbf{r}) = \frac{\partial v_x(\mathbf{r})}{\partial x} + \frac{\partial v_y(\mathbf{r})}{\partial y} + \frac{\partial v_z(\mathbf{r})}{\partial z}
\]

- Turns vector field into scalar field
- Vector function defined at each \( x, y, z \) point \( \mathbf{r} \)
- Change of just \( x \)-component of \( \mathbf{v} \) in \( x \)-direction at point \( \mathbf{r} \)

**Curl** \( \mathbf{\nabla} \times \mathbf{v}(\mathbf{r}) \) (derivative "cross product")

\[
\mathbf{\nabla} \times \mathbf{v}(\mathbf{r}) = \left( \frac{\partial v_z(\mathbf{r})}{\partial y} - \frac{\partial v_y(\mathbf{r})}{\partial z} \right) \mathbf{i} + \left( \frac{\partial v_x(\mathbf{r})}{\partial z} - \frac{\partial v_z(\mathbf{r})}{\partial x} \right) \mathbf{j} + \left( \frac{\partial v_y(\mathbf{r})}{\partial x} - \frac{\partial v_x(\mathbf{r})}{\partial y} \right) \mathbf{k}
\]

- Turns vector field into another vector field
- Vector function defined at each \( x, y, z \) point \( \mathbf{r} \)
- Change of just \( z \)-component of \( \mathbf{v} \) in \( y \)-direction at point \( \mathbf{r} \)

**Vector identities**

\[
\mathbf{\nabla} \times \mathbf{\nabla} s = 0 \quad \text{curl of the gradient of any scalar field is zero}
\]

\[
\mathbf{\nabla} \cdot (\mathbf{\nabla} \mathbf{A}) = 0 \quad \text{divergence of the curl of any vector field is zero}
\]

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}
\]
**POTENTIAL ($\Phi$), ELECTRIC FIELD ($\nabla \Phi$), CSD ($\nabla \cdot (\nabla \Phi) = \nabla^2 \Phi$)**

**Low-frequency field approximation**
- electric fields uncoupled from magnetic ($\nabla \times$; electromagnetic radiation) 
  $\Rightarrow$ pre-Maxwellian approx. (EEG freq's $\ll 1$ MHz)
- calculate electric fields as if magnetic fields don't exist
- calculate magnetic fields strictly from distribution of currents
- ignore capacitative effects, too

Scalar potential, $\Phi$ (what we measure with electrode)

$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$

$\mathbf{E}$ is electric field vector

$\nabla \cdot (\nabla \Phi)$ is Laplacian of $\Phi$ ($= \text{div } \mathbf{E}$)

1. $\mathbf{E}$ defined as force (vector) acting on unit charge at a given point in space (as result of arbitrary distribution of other charges)
2. Current density, $\mathbf{J}$ (true curr. source dens.!) is proportional to $\mathbf{E}$ $\Rightarrow$ still a vector!
3. Ohm's Law; conductivity $\sigma$

$\mathbf{J} = \mathbf{E}$

CSD is Laplacian of $\Phi$ ($= \text{div } \mathbf{E}$)

$\nabla \cdot (-\nabla \Phi) = \text{scalar field} = - \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] = -\nabla^2 \Phi$

3D CSD gold standard (rat PAER paper)

$\Phi$ data $\nabla \rightarrow \nabla \Phi \rightarrow \nabla \cdot (-\nabla \Phi)$

scalar field source/sink movie as function of $t$
1D and 2D Current Source Density Expts.

1D CSD
- Raw, event-related signal relative to ground, \( \frac{\Delta}{\text{time}} \) (eg. skull)
- High pass
- Low pass
- Spikes (upside down extracellular)
- LFP (local field potential)
- Both types of data can be recorded from same electrode

Rationale: CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet
- Assume approx. constant \((\approx 0)\) parallel to cortex

2D CSD
- 2D array of electrodes on pial surface or on scalp

Rationale: all electrodes record along same surface, so assume depth profiles are constant

\(-V^2\) means find spatial (i.e., 1D depth) curvature of potential
- Discrete approx: center \(-\frac{\text{above} + \text{below}}{2}\)
- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies \textbf{sink}!

For scalp recordings, sources and sinks are at the scalp (not a depth line method unless done in 3D)
- Can tell curvature from sign of potential!
  - Concave \(\rightarrow\) sink
  - Convex \(\rightarrow\) source

(Do not confuse \(\nabla^2\) of calc with always a second deriv.)
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukov & Barth, 1998)

CSD - (1)

Layer 4

0 mm

2

50 msec

Source

sink

4

P1

N1

P2

- phase-locked CSD p
  gamma shifts w/ each cycle
**MAXWELL EQUATIONS**

- **Electrostatics, Magnetostatics**
  - Low freq limit
  - Like C&D but times \( \sigma \), conductivity
    - \( \sigma = I/V \) (resistivity)
    - \( \sigma = V/I \) (conductivity)

- **N.B.** These are all defined at a (every) point in space

\[
\nabla \cdot (\sigma \nabla \Phi) = \nabla \cdot \vec{J}
\]

- Conductivity, constant (\( \sigma \))
  - Tensor constant if inhomogeneous in different directions
  - Gradient of scalar potential (what we measure inversely at each point)

\[
\nabla \cdot \vec{B} = 0
\]

- Impressed currents
  - Due to ionic flow
  - That appear out of nowhere
  - (Oeuf, batteries)

\[
\nabla \times \vec{B} = \mu_0 (\vec{J} - \sigma \nabla \Phi)
\]

- **Incidentally, this Maxwell equation violates a**
  - **Divergence**
  - In \( B_z \) in \( x \)
  - Of \( \Phi \) in \( y \)

- **Curl**
  - Field vectors at each point

- **Proper**
  - Magnetic field
  - Permeability
  - Impressed currents
  - Conductivity
  - Gradient of scalar potential (\( -E \))

---

- Propagation of potentials, magnetic fields instantaneous (no capacitance)
- Simultaneous eqns to solve: \( \vec{J} \) are sources, \( \Phi, \vec{B} \) are data
- Linear

Potential(\( \Phi \)) and magnetic fields(\( \vec{B} \)) produced by a weighted sum of two current source distributions are equal to weighted sum of fields produced by each current source distribution by itself:
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[
\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}
\]

(from Nunez, 1981)

\[
\vec{B} = \nabla \times \vec{A}
\]

vector potential

\[
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}
\]

magnetic field

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\]

take \(\nabla \times\) of both sides

use \(\vec{B} = \mu \vec{H}\)

substitute this into \(\nabla \times \vec{E}\)

\[
\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right)
\]

If linear in conductivity and dielectric, too, and fields periodic w/\(\Gamma\)

\[
\nabla \times \nabla \times \vec{E} = -2\pi i \mu (\sigma + 2\pi i \epsilon) \vec{E}
\]

to neglect:

\[
\frac{2\pi \mu (\sigma + 2\pi i \epsilon) |\vec{E}|}{|\nabla \times \nabla \times \vec{E}|} \ll 1
\]

1) \(|\nabla \times \nabla \times \vec{E}| \approx |\vec{E}|/L^2\) where \(L\) is dept over which \(\vec{E}\) varies significantly
2) \(\mu \& \text{ tissue similar to empty space}\)
3) Assume conservative (large) \(\sigma\), dielectric unit, and EEG freq

\(\epsilon\) number is about \(10^{-6}\) \(\rightarrow\) small
MONPOLE, DIPOLE FORWARD SOL’N

\[ \Phi_1 = \frac{S}{4\pi \sigma r} \]

potential recorded for source monopole
distance, source to measuring point (= || \hat{r} ||)

\[ \Phi_2 = \frac{S}{4\pi \sigma} \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \]
potential recorded for source-sink pair ("far field")
dipole source to measuring distance
dipole sink to measuring distance

\[ \vec{B}_2 \approx \frac{\mu_0}{4\pi} \frac{S \hat{d} \times \hat{r}}{r^3}, \quad r \gg d \]

approximations for "far enough away" measurements (subtracting two \( \frac{1}{r^2} \)’s gives inverse square)
dipole strength

**Since \( \hat{r} \) also in numerator, this now inverse square

\[ \vec{B}_2 \approx \frac{\mu_0}{4\pi} \frac{S \hat{d} \times \hat{r}}{r^3}, \quad r \gg d \]

N.B.: both assume inside infinite isotropic conductor

\[ \Phi_i(t) = e_i s(t) \]
electrode gain or source strength

\[ \vec{b}_i(t) = m_i s(t) \]

Linear superposition with fixed electrodes and sensors

**Since \( \hat{r} \) also in numerator, this now inverse square

\[ X(t) = \sum_j g_{ij} s_j(t) \]
electric measures gain or strength

\[ \vec{X}(t) = \sum_j g_{ij} \vec{s}_j(t) \rightarrow \vec{X}(t) = G \vec{s}(t) \]
Forward Solution

- well-posed (one answer)
- linear: \( 6(A) + 6(B) = 6(A+B) \)
- approximations due to unknown electrical properties of head

3-shell spherical analytic

- skull/brain conductivity
- "smearing" (cf. cable theory)

3-shell boundary element

- arbitrary shape
- homogeneous conductivity

solution = infinite homogeneous + matrix of correction factors

Finite element

[most general computational intensive w small grid]
[many unknown parameters to estimate]

remember, we only need to able to calc. weight for each dipole/ electrode pair independently

for magnetic, only need one shell b/c currents thru skin/skull too small to make sig. \( \mathbf{B} \)
**Forward Sol'n**

\[ v_i = \sum_j \varepsilon_{ij} s_j + \eta_i \]

**Matrix Form**

\[
\begin{bmatrix}
\mathbf{v}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{E}
\end{bmatrix}\mathbf{s} + \mathbf{n}
\]

- **E** fixed across time
- **v, s, n** vary
- Lower case bold \(\rightarrow\) vector
- Upper case bold \(\rightarrow\) matrix

**Electric Recordings**

\[
\begin{bmatrix}
\mathbf{V}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{E}
\end{bmatrix}\mathbf{s} + \mathbf{n}
\]

**Magnetic Recordings**

\[
\begin{bmatrix}
\mathbf{m}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{B}
\end{bmatrix}\mathbf{s} + \mathbf{n}
\]

**Note:** Only one current source for each column in the **E+B** matrix!
WHY LOCALIZE?

- most of ERP literature based (instead) on temporal "components"

but:  
1) underly local cortical generators (from microelectrode LFP)
   - extended in time (400 ms)
   - multiphasic in every cortical area
   - temporally non-static depending on stimulus
     - e.g. simple contrast-brightness changes modulate retinal delay by 50 ms!

2) thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels

3) stimulus manipulations will change temporal overlap
   - may cause "component" peak to disappear without changing cortical areas being activated

4) verified by intra cortical LFP/CSD (Schroeder et al, 1998)

\[ \text{LFPs from approx. layer 4 in cortex ("input layer")} \]

psychologists now identify a few temporal "component" peaks...

but each one comes from every one of these cortical areas!!

- by contrast, the spatial signature of the signal from one cortical area is static — a better area-based component

- origin of "components" (easier to record more temporal points)
  (EEG started with few electrodes, many time points)

  - easier to "paste" high level psychological functions onto a few waveform deflections
Derivation of Ill-posed Inverse

(from Dale & Sereno, 1993)

\[ x = As + n \]

\( A = \text{forward sphl matrix (E+B)} \)

\( s = \text{source vector} \)

\( n = \text{sensor noise vector} \)

\[ \text{solve for inverse operator} \]

\[ \text{operator} \]

\( \text{probability that rand. var. value is k} \)

\[ \text{expectation: } \sum_k P_k k \]

\[ \text{assume } n, s \text{ normal, zero-mean } w \text{ corresponding covar. matrices } C, R \]

\[ \text{Err}_w = \langle \| W(As+n) - s \| ^2 \rangle \]

\[ = \langle \| (WA-I)s + Wn \| ^2 \rangle \]

\[ = \langle \| Ms + Wn \| ^2 \rangle \]

\[ = \langle \| Ms \| ^2 \rangle + \langle \| Wn \| ^2 \rangle \]

\[ = \text{trace is noise variance (already squared)} \]

\[ = \text{trace is sum of diag elements} \]

\[ \text{[re-expand]} = \text{trace } (WARA^TW - RATA^TW - WAR + R) + \text{trace } (WCW^T) \]

\[ \text{Explicitly minimize by taking derivative wrt } W, \text{ set to zero, solve for } W \]

\[ 0 = 2WARA^T - 2RATA^T + 2WC \]

\[ W = \frac{RATA^T}{WARA^T + WC} = RA^T \]

\[ W = RA^T(AATA^T+C)^{-1} \]

\( W \) is inverse solution operator:

\[ \text{sensor inputs: } \begin{bmatrix} W \end{bmatrix} \]

\( W \) is equivalent to minimum norm and Tikhonov regularized inverse if \( C, R \) are proportional to identity matrix (i.e., sensor noise & sources independent and equal variance).
INVERSE Sifting (2)

\[
W = RA^T(ARA^T + C)^{-1}
\]

- With liberal priors, sources uncorrelated (R)
- Noise uncorrelated (C)

\[ \Rightarrow \text{"minimum norm" solution} \]
\[ \text{(find } \hat{\beta} \text{ w/ smallest norm = } 1151) \]

- The minimum norm solution appropriately downplays deeper (=weaker scalp signal) sources since these are more likely to fall into the noise floor.

- "Problems" of minimum norm:
  - Deeper sources get displaced to the surface.
  - Small superficial sources "win" because of approx. inverse square form of true solution.
  - Smaller norm w/ equivalent fit/error.
  - Can't fix by increasing priors of deep sources!!
  - That will give deep sources given noise as input!!
Inverse solutions to ill-posed compared

\[ s = Wx \]

\[ s = \begin{bmatrix} s_1 & s_2 & \ldots & s_n \end{bmatrix} \]

Sensor data

\[ W \]

Least squares solution

Smallest of infinitely many alternate solutions

\[ W = (A^T A + R)^{-1} A^T c \]

Alternate, algebraically equivalent Bayesian derivation (w/ bigger inverses!)

\[ W = \begin{bmatrix} W \end{bmatrix} = \begin{bmatrix} A^T[c] \end{bmatrix} \begin{bmatrix} A \end{bmatrix} + \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} c \end{bmatrix} \]

Both square in # of sources (large)

Easier inverse

Square in # of sensors (small)
PROBLEMS W/ SURFACE NORMAL

- Since nearby points on a surface often have different orientation, surface normal constraint can help (since fwd soln A,B very different)

- But, since point spread function typically extends across sulci, artificial sign reversals occur

- Solutions

  1) Ignore sign $\implies$ saves useful orientation info!

  2) Solve onto 3 orthogonal dipoles at each critical point instead of a single oriented dipole

    $\implies$ more appropriate when averaging across subjects, since detailed variations vary a lot

    $\implies$ also, fills in bottom of sulci (else unsigned stripes)
FMRI Constrained Inverse

- insert FMRI values for R_{ii}'s
- but still allow other sites to have non-zero R_{ii}'s
- pathologies occur if solution restricted completely to FMRI points by setting non-FMRI R_{ii}'s to zero set to small number instead!

- this allows extracting time course from sources visible in EEG/MEG and FMRI

- N.B.: sources that are only visible in EEG/MEG will be dispersed to small distributed values at a large number of vertices
  visible in both EEG/MEG and FMRI
  visible only in EEG/MEG and not FMRI distributed at small amplitude across many vertices
**Noise Sensitivity Normalization**

\[
\text{Forward: } x = Ax \quad \text{well-posed} \quad \text{(Liu, Dale, and Belliveau, 2002)}
\]

\[
\text{Inverse: } s = Wx \quad \text{ill-posed}
\]

\[
\text{Solve: } x = As + n \quad \text{for } s
\]

\[
W = R A^T (A R A^T + C)^{-1}
\]

- Multiply inverse operator by noise sensitivity matrix, \( D \) (diagonal)

\[
D_{ii} = \text{diag} \left( \sqrt{WCW^T} \right)
\]

\[
w_{\text{norm}} = Dw
\]

\[
s_{i}^{\text{norm}} = (w_{\text{norm}} x)_i = (Dw x)_i = \frac{W_i x}{\sqrt{(WCW^T)_i}}
\]

If assume Gaussian white noise,

- Noise covariance, \( C \), is multiple of \( I \), so

\[
w_i^{\text{norm}} = \frac{W_i^{\text{orig}}}{\| W_i^{\text{orig}} \|}
\]

\[
s_i = \frac{W_i^{\text{orig}} \cdot \bar{x}}{\| W_i^{\text{orig}} \|}
\]

\[
\begin{bmatrix}
\frac{\text{sum of squares}}{\text{one row}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{one row}
\end{bmatrix} = \begin{bmatrix}
\text{one source}
\end{bmatrix}
\]

\[
S_i = \frac{W_i^{\text{orig}} \cdot \bar{x}}{\| W_i^{\text{orig}} \|}
\]

\[
\begin{bmatrix}
\text{sum of rows}
\end{bmatrix} = \begin{bmatrix}
\text{one row}
\end{bmatrix} \cdot \begin{bmatrix}
W
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{one source}
\end{bmatrix} = \begin{bmatrix}
\text{inverse solution coefficients}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{one row}
\end{bmatrix} \cdot \begin{bmatrix}
W
\end{bmatrix}
\]

\[
\text{scale by norm of this row}
\]

\[
\text{i.e., if inverse solution for source is big (e.g., deep source), noise norm inverse for that source reduced by scaling}
\]

\[
\text{inverse solution coefficient for one source}
\]

\[
\| W_i \| = \sqrt{(WCW^T)_i}
\]

\[
\text{each entry is sum of squares}
\]

\[
\text{scale by norm of this row}
\]

\[
\text{i.e., if inverse solution for source is big (e.g., deep source), noise norm inverse for that source reduced by scaling}
\]
**Noise Sensitivity Normalization (2)**

Shallow source (unit strength)  

**Input big**  
**Input small**

Deep source (unit strength)  

**Input smaller because of minimum norm**

\[
S_i = \frac{\hat{W}_i \cdot \hat{X}}{||\hat{W}_i||}
\]

- Effect on inverse solution \(\rightarrow\) more like significance \(\neq\) actual power
- Effect on point-spread function is to equalize shallow & deep
  - Shallow spread out more than min norm
  - Deep shrunk to same as shallow

Point-spread functions
  
  Noise \(\rightarrow\) normed

Point-spread
  
  Noise \(\rightarrow\) normed

Therefore, fixed estimate increased relative to shallow
**Conclusions**

- More EEG or more MEG better.
- EEG better than MEG (cf. radial) (EEG far less accurate currently).
- Biggest gain from adding small # EEG (a MEG) (e.g. 20) to many MEG (a EEG) (e.g. 150).
- Easier to add many MEG, so: optimal < 30 EEG
  < 300 MEG.

**EEG/MEG forward-solution-scaling-factor error**  
causes → more cross talk

- **Cross talk**, columns of resolution matrix, WA
  
  \[
  \xi_{ij}^2 = \left(\frac{(W\tilde{A})_{ij}}{(W\tilde{A})_{ii}}\right)^2 = \frac{|\text{W}_i \tilde{A}_j|^2}{|\text{W}_i \tilde{A}_i|^2}
  \]

- Avg. cross talk
  
  \[
  \text{ACM}_i = \frac{\sum_j \xi_{ij}^2}{j}
  \]

- **Point spread**, rows of resolution matrix, WA
  
  \[
  \rho_{ij}^2 = \left(\frac{(W\tilde{A})_{ji}}{(W\tilde{A})_{ii}}\right)^2 = \frac{|\text{W}_i \tilde{A}_j|^2}{|\text{W}_i \tilde{A}_i|^2}
  \]

- Avg. point spread
  
  \[
  \text{APSF}_i = \frac{\sum_j \rho_{ij}^2}{j}
  \]

- PSF & cross talk maps identical for standard inverse (WA the "resolution matrix" is symmetric)
- PSF affected by noise-normalized; cross talk same (DWA not symmetric)
**Music**

(from Dale & Sereno, 1993) (ct. Mosher & Leahy)

- Using sensor covariance

\[
D = \langle xx^T \rangle = \sigma^2 I + \sum_i \sum_j \sigma_i \sigma_j C_{ij} A_i A_j^T
\]

\[
\mathbf{D} = \mathbf{U} \Lambda \mathbf{U}^T = \begin{bmatrix}
\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n
\end{bmatrix} \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix} \begin{bmatrix}
\mathbf{u}_1^T \\
\mathbf{u}_2^T \\
\vdots \\
\mathbf{u}_n^T
\end{bmatrix}
\]

- Find most significant spatial patterns in sensors over time

Project forward solutions onto these spatial patterns

("project" = dot prod = similarity) for each point in brain

\[
\mathbf{\hat{z}}_i = A_i \mathbf{U} \Lambda \mathbf{U}^T A_i^T 
\]

= big single number if forward solution looks like \( \mathbf{U} \)
(2) How to weight the minimum norm inverse

\[ R_{ii} \approx \frac{A_i^T A_i}{A_i^T U \Lambda U^T A_i} \]

Total \[ R \]

\[ W = R A^T (A R A^T + C)^{-1} \]

cf.

Like parallel resistance:

\[ R_{\text{parallel}} = \frac{1}{1/R_1 + 1/R_2 + \ldots} \]

So any low resistance (\( \xi_i \)) decreases overall resistance (small \( R_{ii} \))

So, if toward null has appearance like any low eigen-value spectral pattern, it gets devalued
- how it works

one time point

multiple time points

- how it fixes min norm problem

N.B., if two widely separated sources (i.e., diff fard soln's) are highly correlated, MUSIC will eliminate both since no single fard soln will look like that 2-separated dipole pattern (e.g., L/R A-I)

le various "dual MUSIC" hacks possible